GES-207-4 — Mathematical Methods of Physics

National Institute for Space Research (INPE)
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Lecture Notes 16

Legendre Polynomials and Spherical Harmonics

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Learning goals and study scope. This self-contained lecture derives Legendre's equation from Laplace's equation in spherical coordinates and solves it by a power series. Regularity at both poles selects a discrete set of eigenvalues and produces the Legendre polynomials. We then develop Rodrigues' formula, the generating function, recurrence relations, orthogonality, and Legendre expansions. Associated Legendre functions and spherical harmonics extend the construction to general angular dependence, with the geomagnetic potential as the main space-physics application.

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1. Origin in Laplace's Equation

In a region without sources, a scalar potential V satisfies Laplace's equation

$$\nabla^2 V = 0. \tag{1}$$

For an axisymmetric potential in spherical coordinates, there is no dependence on the azimuthal angle φ, and

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) = 0. \tag{2}$$

Seek a separated solution

$$V(r,\theta) = R(r)\Theta(\theta).$$

Substitution into (2), division by $R\Theta$, and multiplication by r^2 give

$$\frac{1}{R}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2R'\right) + \frac{1}{\Theta\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\,\Theta'\right) = 0.$$

The first term depends only on r and the second only on θ. Their sum can vanish for all r and θ only if both are constants. Choose the separation constant as $\ell(\ell+1)$:

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2R'\right) - \ell(\ell+1)R = 0, \tag{3}$$

$$\frac{1}{\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\,\Theta'\right) + \ell(\ell+1)\Theta = 0. \tag{4}$$

The radial equation is the Euler equation studied in Lecture 14:

$$r^2R'' + 2rR' - \ell(\ell+1)R = 0, \tag{5}$$

with solution

$$R(r) = Ar^{\ell} + Br^{-(\ell+1)}. \tag{6}$$

The new problem is the angular equation.

1.1. From polar angle to the Legendre variable

Set

$$x = \cos\theta, \qquad -1 \le x \le 1. \tag{7}$$

Since $\mathrm{d}x/\,\mathrm{d}\theta = -\sin\theta$,

$$\frac{\mathrm{d}\Theta}{\mathrm{d}\theta} = -\sin\theta\frac{\mathrm{d}\Theta}{\mathrm{d}x}.$$

Consequently,

$$\frac{1}{\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\frac{\mathrm{d}\Theta}{\mathrm{d}\theta}\right) = \frac{\mathrm{d}}{\mathrm{d}x}\left[(1-x^2)\frac{\mathrm{d}\Theta}{\mathrm{d}x}\right].$$

Equation (4) becomes

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}x}\left[(1-x^2)y'\right] + \ell(\ell+1)y = 0,} \tag{8}$$

or

$$\boxed{(1-x^2)y'' - 2xy' + \ell(\ell+1)y = 0.} \tag{9}$$

This is Legendre's differential equation.

2. Ordinary and Singular Points

After division by $1-x^2$,

$$y'' - \frac{2x}{1-x^2}y' + \frac{\ell(\ell+1)}{1-x^2}y = 0.$$

The point $x = 0$ is ordinary, so an ordinary power series can be used there. The endpoints $x = \pm 1$ are singular because $1 - x^2 = 0$. Both are regular singular points.

These endpoints correspond to the north and south poles of the sphere. A physical potential must remain finite at both. The endpoint requirement will restrict the allowed values of the separation parameter.

> The symbols "pole of the sphere" and "pole of a function" have different meanings. Here $\theta = 0$ and $\theta = \pi$ are geographic or geometric poles. They are regular singular points of the differential equation, not necessarily poles in the complex-analysis sense.

3. Power-Series Solution

Expand about the ordinary point $x = 0$:

$$y(x) = \sum_{k=0}^{\infty} a_k x^k. \tag{10}$$

Then

$$y' = \sum_{k=0}^{\infty}(k+1)a_{k+1}x^k,$$
$$y'' = \sum_{k=0}^{\infty}(k+2)(k+1)a_{k+2}x^k.$$

The terms multiplied by x and x^2 are

$$2xy' = \sum_{k=0}^{\infty} 2ka_k x^k,$$
$$x^2y'' = \sum_{k=0}^{\infty} k(k-1)a_k x^k,$$

where the formulas also hold for $k = 0, 1$ because the corresponding factors vanish.

Substitution into (9) gives

$$\sum_{k=0}^{\infty} \{(k+2)(k+1)a_{k+2} + [\ell(\ell+1) - k(k+1)]\, a_k\}\, x^k = 0.$$

Thus

$$\boxed{a_{k+2} = \frac{k(k+1) - \ell(\ell+1)}{(k+1)(k+2)} a_k.} \tag{11}$$

The recurrence advances by two. The coefficient a_0 generates an even series, while a_1 generates an odd series. These are the two independent local solutions near the origin.

4. Truncation and Discrete Eigenvalues

For a generic value of ℓ, the series extends indefinitely and typically fails to remain finite at both $x = 1$ and $x = -1$. A globally regular angular solution appears when

$$\ell = n, \qquad n = 0, 1, 2, \ldots.$$

Then the numerator in (11) vanishes at $k = n$:

$$n(n+1) - n(n+1) = 0.$$

The coefficient chain with the same parity as n terminates. The result is a polynomial of degree n.

Legendre polynomial

For each nonnegative integer n, the Legendre polynomial $P_n(x)$ is the polynomial solution of

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[(1-x^2)P_n'(x)\right] + n(n+1)P_n(x) = 0 \tag{12}$$

normalized by

$$\boxed{P_n(1) = 1.} \tag{13}$$

The regularity condition has turned the continuous separation parameter into the discrete eigenvalues

$$\boxed{\lambda_n = n(n+1).} \tag{14}$$

This is a standard pattern in boundary-value problems: regularity or boundary conditions select discrete eigenvalues.

Common pitfall

When $\ell = n$, only the coefficient chain with the parity of n truncates to P_n. The other independent local solution does not become a second polynomial and is singular at an endpoint. A second-order equation still has two local solutions, but the physical conditions select one.

5. The First Legendre Polynomials

Using the recurrence and the normalization $P_n(1) = 1$ gives

$$\begin{aligned} P_0(x) &= 1, \\ P_1(x) &= x, \\ P_2(x) &= \frac{1}{2}(3x^2 - 1), \\ P_3(x) &= \frac{1}{2}(5x^3 - 3x), \\ P_4(x) &= \frac{1}{8}(35x^4 - 30x^2 + 3), \\ P_5(x) &= \frac{1}{8}(63x^5 - 70x^3 + 15x). \end{aligned}$$

They satisfy

$$P_n(-x) = (-1)^n P_n(x), \tag{15}$$

so even n gives an even polynomial and odd n an odd polynomial. At the endpoints,

$$P_n(1) = 1, \qquad P_n(-1) = (-1)^n. \tag{16}$$

Example 16.1: checking P_2 directly

For

$$P_2(x) = \frac{1}{2}(3x^2 - 1),$$

we have $P_2' = 3x$ and $P_2'' = 3$. Substitution into Legendre's equation with $n = 2$ gives

$$\begin{aligned} (1 - x^2)P_2'' - 2xP_2' + 6P_2 &= 3(1 - x^2) - 6x^2 + 3(3x^2 - 1) \\ &= 0. \end{aligned}$$

Also $P_2(1) = 1$, as required.

6. Rodrigues' Formula

Rodrigues' formula

The normalized Legendre polynomial of degree n is

$$\boxed{P_n(x) = \frac{1}{2^n n!}\frac{\mathrm{d}^n}{\mathrm{d}x^n}(x^2-1)^n.} \tag{17}$$

For $n = 2$,

$$(x^2-1)^2 = x^4 - 2x^2 + 1,$$

so

$$P_2(x) = \frac{1}{8}\frac{\mathrm{d}^2}{\mathrm{d}x^2}(x^4 - 2x^2 + 1) = \frac{1}{8}(12x^2 - 4) = \frac{1}{2}(3x^2 - 1).$$

For $n = 3$,

$$P_3(x) = \frac{1}{48}\frac{\mathrm{d}^3}{\mathrm{d}x^3}(x^2-1)^3 = \frac{1}{2}(5x^3 - 3x).$$

Rodrigues' formula makes the parity and polynomial degree immediate and is especially useful for proving orthogonality.

7. The Generating Function

Generating function

The Legendre polynomials are generated by

$$\boxed{\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x)t^n, \qquad |t| < 1.} \tag{18}$$

Expanding the left side in powers of t reproduces $P_0, P_1, P_2, \ldots$. Differentiating the generating function and comparing coefficients yields useful recurrence and derivative identities.

The generating function has a direct physical interpretation. If $r > r'$ and γ is the angle between $\boldsymbol{r}$ and $\boldsymbol{r}'$, then

$$\begin{aligned}\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|} &= \frac{1}{r}\frac{1}{\sqrt{1-2(r'/r)\cos\gamma + (r'/r)^2}} \\ &= \boxed{\sum_{n=0}^{\infty}\frac{r'^n}{r^{n+1}}P_n(\cos\gamma)}.\end{aligned}$$

This is the multipole expansion of the Coulomb or gravitational kernel.

8. Recurrence and Derivative Identities

The most common three-term recurrence is

$$\boxed{(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x).} \tag{19}$$

Starting from $P_0 = 1$ and $P_1 = x$, it generates all higher polynomials without repeated differentiation.

Another useful identity is

$$\boxed{(1-x^2)P_n'(x) = n\left[P_{n-1}(x) - xP_n(x)\right].} \tag{20}$$

These relations are convenient in numerical calculations and in manipulating field components derived from a scalar potential.

Example 16.2: generating P_3 by recurrence

Set $n = 2$ in (19):

$$3P_3 = 5xP_2 - 2P_1.$$

Using $P_2 = (3x^2-1)/2$ and $P_1 = x$,

$$P_3 = \frac{1}{3}\left[\frac{5}{2}x(3x^2-1) - 2x\right] = \frac{1}{2}(5x^3 - 3x).$$

9. Orthogonality

Legendre's equation has Sturm–Liouville form

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[(1-x^2)P_n'\right] + n(n+1)P_n = 0.$$

Let P_m and P_n satisfy the equations with distinct degrees. Multiply the n equation by P_m, the m equation by P_n, and subtract:

$$P_m\frac{\mathrm{d}}{\mathrm{d}x}\left[(1-x^2)P_n'\right] - P_n\frac{\mathrm{d}}{\mathrm{d}x}\left[(1-x^2)P_m'\right] + \left[n(n+1) - m(m+1)\right]P_mP_n = 0.$$

Integrate from -1 to 1. The derivative terms reduce to the boundary term

$$\Big[(1-x^2)(P_mP_n' - P_nP_m')\Big]_{-1}^{1} = 0$$

because $1-x^2$ vanishes at both endpoints and the polynomials remain finite. For $m \neq n$, the eigenvalues differ, so

$$\boxed{\int_{-1}^{1} P_m(x)P_n(x)\,\mathrm{d}x = 0, \qquad m \neq n.} \tag{21}$$

The normalization integral is

$$\boxed{\int_{-1}^{1}[P_n(x)]^2\,\mathrm{d}x = \frac{2}{2n+1}.} \tag{22}$$

Together,

$$\boxed{\int_{-1}^{1} P_m(x)P_n(x)\,\mathrm{d}x = \frac{2}{2n+1}\delta_{mn}.} \tag{23}$$

10. Legendre-Series Expansions

For a suitable function f on $[-1,1]$, write

$$f(x) = \sum_{n=0}^{\infty} c_n P_n(x). \tag{24}$$

Multiply by P_m and integrate. Orthogonality removes every term except $n = m$:

$$\boxed{c_n = \frac{2n+1}{2}\int_{-1}^{1} f(x)P_n(x)\,\mathrm{d}x.} \tag{25}$$

This is the analogue of projection onto an orthogonal basis from Lecture 3.

Example 16.3: expanding x^2

Since

$$P_0 = 1, \qquad P_2 = \frac{1}{2}(3x^2-1),$$

solve the second equation for x^2:

$$\boxed{x^2 = \frac{1}{3}P_0(x) + \frac{2}{3}P_2(x).} \tag{26}$$

No odd polynomial appears because x^2 is even. Direct use of (25) gives the same coefficients.

11. Associated Legendre Functions

When the potential depends on the azimuthal angle, separation takes the form

$$V(r,\theta,\varphi) = R(r)\Theta(\theta)\Phi(\varphi).$$

Single-valuedness in φ gives

$$\Phi(\varphi) = e^{im\varphi}, \qquad m \in \mathbb{Z}.$$

The polar equation becomes the associated Legendre equation

$$(1-x^2)y'' - 2xy' + \left[\ell(\ell+1) - \frac{m^2}{1-x^2}\right]y = 0. \tag{27}$$

For $0 \le m \le \ell$, define

$$\boxed{P_\ell^m(x) = (-1)^m (1-x^2)^{m/2} \frac{\mathrm{d}^m}{\mathrm{d}x^m} P_\ell(x).} \tag{28}$$

The factor $(-1)^m$ is the Condon–Shortley phase convention. Some references omit it, so conventions must be checked when comparing formulas.

12. Spherical Harmonics

The angular eigenfunctions on the sphere are

$$\boxed{Y_\ell^m(\theta,\varphi) = N_{\ell m} P_\ell^m(\cos\theta) e^{im\varphi},} \tag{29}$$

where one common normalization is

$$N_{\ell m} = \sqrt{\frac{2\ell+1}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}}. \tag{30}$$

They satisfy

$$\int_0^{2\pi}\int_0^{\pi} Y_\ell^m(\theta,\varphi)^* Y_{\ell'}^{m'}(\theta,\varphi)\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi = \delta_{\ell\ell'}\delta_{mm'}. \tag{31}$$

The integer ℓ controls the total angular degree, while m controls the azimuthal variation. For each ℓ, there are $2\ell+1$ values $m = -\ell, \dots, \ell$.

13. Application to the Geomagnetic Field

In a current-free region, the magnetic field can be written as

$$\boldsymbol{B} = -\nabla V, \qquad \nabla^2 V = 0.$$

A standard internal-source spherical-harmonic representation is

$$\boxed{V(r,\theta,\varphi) = a\sum_{n=1}^{N}\left(\frac{a}{r}\right)^{n+1}\sum_{m=0}^{n}\left[g_n^m\cos(m\varphi) + h_n^m\sin(m\varphi)\right]\overline{P}_n^m(\cos\theta),} \tag{32}$$

where a is a reference Earth radius, g_n^m and h_n^m are Gauss coefficients, and $\overline{P}_n^m$ denotes the normalization convention used by the geomagnetic model.

The degree $n = 1$ describes the dipole contribution. Degrees $n = 2, 3, \dots$ describe progressively finer angular structure such as quadrupole and octupole components. The radial factor $(a/r)^{n+1}$ follows from the decaying Euler solution $r^{-(n+1)}$ derived in Lecture 14.

> Associated Legendre functions appear with several normalizations in physics, geodesy, and geomagnetism. The coefficients g_n^m, h_n^m depend on the chosen normalization. A table of coefficients and a formula for P_n^m must use the same convention.

14. Method Checklist and Common Errors

When working with Legendre polynomials:

1. identify the physical domain $-1 \leq x \leq 1$ and the endpoint regularity requirement;
2. derive the coefficient recurrence with the powers aligned;
3. distinguish the local two-dimensional solution space from the single solution regular at both endpoints;
4. state the normalization, usually $P_n(1) = 1$;
5. use orthogonality to calculate expansion coefficients;
6. check phase and normalization conventions for associated functions and spherical harmonics.

Common pitfall

The eigenvalue is $n(n+1)$, not simply n^2. Also, P_n is orthogonal but not normalized to unit norm: its squared norm is $2/(2n+1)$.

15. Summary

- Legendre's equation is the angular equation obtained by separating Laplace's equation in spherical coordinates.
- Regularity at both spherical poles selects the eigenvalues $n(n+1)$ and the polynomial solutions P_n.
- Rodrigues' formula, the generating function, and the three-term recurrence provide complementary ways to construct the polynomials.
- Orthogonality makes the Legendre polynomials a natural basis on $[-1, 1]$.
- Associated Legendre functions and spherical harmonics describe general angular dependence on the sphere.
- Geomagnetic potential models combine spherical harmonics with the radial power laws obtained from the Euler equation.