GES-207-4 — Mathematical Methods of Physics

National Institute for Space Research (INPE)
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Lecture Notes 15

Power Series, the Frobenius Method, and the Indicial Equation

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Learning goals and study scope. This self-contained lecture develops local series solutions of second-order linear differential equations. We begin at ordinary points, where an ordinary power series is sufficient, and then classify singular points. At a regular singular point, the Frobenius method allows a noninteger leading power determined by the indicial equation. Bessel's equation provides the main worked example and connects the method to waves and fields in cylindrical geometry.

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1. Local Series as a Solution Method

Consider a homogeneous second-order linear equation in normalized form,

$$y'' + P(x)y' + Q(x)y = 0. \tag{1}$$

When elementary closed forms are unavailable, a local series can determine the solution coefficient by coefficient. The method replaces differentiation of an unknown function by algebraic relations among numbers.

Near a point x_0, begin with

$$\boxed{y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n.} \tag{2}$$

Inside its radius of convergence, this series may be differentiated term by term:

$$y'(x) = \sum_{n=1}^{\infty} n a_n (x - x_0)^{n-1}, \tag{3}$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n (x - x_0)^{n-2}. \tag{4}$$

The first two coefficients have an immediate meaning:

$$a_0 = y(x_0), \qquad a_1 = y'(x_0). \tag{5}$$

Thus the two free coefficients encode the two initial conditions expected for a second-order equation.

2. Index Shifts and Recurrence Relations

After substituting the series, every sum must be expressed with the same power of $(x - x_0)$ before coefficients can be compared. For an expansion about zero,

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n. \tag{6}$$

The change of index does not change the terms represented by the series. The new lower limit and every occurrence of the index must be adjusted together.

The standard workflow is:

1. write y, y', and y'' as series about the chosen point;
2. substitute them into the differential equation;

3. shift indices until the powers match;
4. separate any exceptional low-order terms;
5. equate the coefficient of each power to zero;
6. use the resulting recurrence to generate the coefficients.

Common pitfall

One cannot add or compare two series merely because they use the same symbol n. Their powers must match. For example, a coefficient multiplying x^{n-2} cannot be combined with one multiplying x^n until an index shift is made.

3. A Familiar Check: Recovering Sine and Cosine

Apply the series method to

$$y'' + y = 0. \tag{7}$$

Take $y = \sum_{n=0}^{\infty} a_n x^n$. Then

$$y'' = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n.$$

Substitution gives

$$\sum_{n=0}^{\infty} \left[(n+2)(n+1)a_{n+2} + a_n\right] x^n = 0.$$

A power series is identically zero only if every coefficient vanishes, so

$$\boxed{a_{n+2} = -\frac{a_n}{(n+2)(n+1)}.} \tag{8}$$

The recurrence advances by two, so the even and odd coefficients form two independent chains. Starting from a_0,

$$a_2 = -\frac{a_0}{2!}, \qquad a_4 = \frac{a_0}{4!}, \qquad a_6 = -\frac{a_0}{6!}, \dots$$

Starting from a_1,

$$a_3 = -\frac{a_1}{3!}, \qquad a_5 = \frac{a_1}{5!}, \qquad a_7 = -\frac{a_1}{7!}, \dots$$

Therefore

$$\begin{aligned} y(x) &= a_0 \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right) + a_1 \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right) \\ &= \boxed{a_0 \cos x + a_1 \sin x}. \end{aligned}$$

The calculation verifies that the series method reproduces the expected elementary solution.

4. A Genuinely Variable Coefficient

Example 15.1: a series for $y'' - xy = 0$

Consider

$$y'' - xy = 0. \tag{9}$$

With $y = \sum_{n=0}^{\infty} a_n x^n$,

$$y'' = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n,$$

while

$$xy = \sum_{n=0}^{\infty} a_n x^{n+1} = \sum_{n=1}^{\infty} a_{n-1} x^n.$$

The coefficient of x^0 gives

$$2a_2 = 0, \qquad a_2 = 0.$$

For $n \geq 1$,

$$\boxed{a_{n+2} = \frac{a_{n-1}}{(n+2)(n+1)}}. \tag{10}$$

This recurrence advances by three. The first coefficients are

$$a_3 = \frac{a_0}{6}, \qquad a_4 = \frac{a_1}{12}, \qquad a_5 = 0,$$
$$a_6 = \frac{a_0}{180}, \qquad a_7 = \frac{a_1}{504}, \qquad a_8 = 0.$$

Thus

$$y(x) = a_0 \left(1 + \frac{x^3}{6} + \frac{x^6}{180} + \cdots\right) + a_1 \left(x + \frac{x^4}{12} + \frac{x^7}{504} + \cdots\right).$$

The free values $a_0 = y(0)$ and $a_1 = y'(0)$ generate two independent solutions. Equation (9) is the Airy equation with the sign convention $y'' - xy = 0$.

5. Ordinary Points and Convergence

Ordinary point

For the normalized equation (1), a point x_0 is an *ordinary point* if $P(x)$ and $Q(x)$ are analytic at x_0. Otherwise, x_0 is a singular point.

At an ordinary point, the equation has two linearly independent solutions that are analytic near x_0. An ordinary power series is therefore the appropriate local form.

The radius of convergence extends at least to the nearest point in the complex plane

where the normalized coefficient functions cease to be analytic. This statement is inherently complex even when the original problem is posed on the real axis.

Example 15.2: the nearest complex singularity

Consider

$$(1+x^2)y'' + xy' + y = 0$$

expanded about $x_0 = 0$. The normalized coefficients are

$$P(x) = \frac{x}{1+x^2}, \qquad Q(x) = \frac{1}{1+x^2}.$$

They are analytic on the real axis, but have complex singularities at $x = \pm i$. Their distance from the origin is one. Therefore the power-series solution about zero is guaranteed only for $|x| < 1$, even though the real coefficients appear smooth for every real x.

A recurrence relation determines formal coefficients. Convergence determines whether the formal series represents a function. For linear equations with analytic coefficients, the local existence theorem supplies convergence near an ordinary point.

6. Singular and Regular Singular Points

A singular point need not be too singular for a controlled series method.

Regular singular point

For

$$y'' + P(x)y' + Q(x)y = 0,$$

a singular point x_0 is *regular singular* if

$$\boxed{(x-x_0)P(x) \quad \text{and} \quad (x-x_0)^2 Q(x)} \tag{11}$$

are analytic at x_0. A singular point that fails this test is irregular.

The condition allows P to have at most a simple pole and Q to have at most a double pole. These controlled divergences permit a solution with a power-law factor multiplying an ordinary series.

Example 15.3: classifying $x = 0$

Consider the following equations.
(a) $x^2y'' + xy' + x^2y = 0$. After division by x^2,

$$P(x) = \frac{1}{x}, \qquad Q(x) = 1.$$

Then $xP = 1$ and $x^2Q = x^2$ are analytic. The origin is regular singular.
(b) $xy'' + y' + y = 0$. After division by x,

$$P(x) = \frac{1}{x}, \qquad Q(x) = \frac{1}{x}.$$

Then $xP = 1$ and $x^2Q = x$ are analytic. The origin is again regular singular.
(c) $x^3y'' + y = 0$. After division by x^3,

$$P(x) = 0, \qquad Q(x) = \frac{1}{x^3}.$$

Now $x^2Q = 1/x$ is not analytic. The origin is irregular singular.

Common pitfall

Always normalize the equation before applying the ordinary-point or regular-singular-point test. Inspecting the coefficients before division by the coefficient of y'' can produce the wrong classification.

7. The Frobenius Method

At a regular singular point, an ordinary Taylor series may be too restrictive. Take $x_0 = 0$ for simplicity and assume

$$\boxed{y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}, \qquad a_0 \neq 0.} \tag{12}$$

The exponent r is not assumed to be an integer. It determines the leading behavior

$$y(x) \sim a_0 x^r \qquad (x \to 0).$$

The derivatives are

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}, \tag{13}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}. \tag{14}$$

Substitute these expressions into the differential equation. The coefficient of the lowest power of x must vanish. This produces the indicial equation. After r is determined, the higher powers produce a recurrence for the a_n.

7.1. General form of the indicial equation

At a regular singular point $x = 0$, define

$$p_0 = \lim_{x \to 0} xP(x), \qquad q_0 = \lim_{x \to 0} x^2 Q(x). \tag{15}$$

The leading terms of the differential equation give

$$\boxed{r(r-1)+p_0 r+q_0=0.} \tag{16}$$

This shortcut determines the indicial roots, but the full substitution is still needed to derive the recurrence.

> The condition $a_0 \neq 0$ is essential. If $a_0 = 0$, the assumed exponent r would not describe the actual leading power; the series could simply be reindexed with a different exponent.

8. Bessel's Equation

Bessel's equation of order ν is

$$\boxed{x^2y''+xy'+(x^2-\nu^2)y=0.} \tag{17}$$

After division by x^2,

$$y''+\frac{1}{x}y'+\left(1-\frac{\nu^2}{x^2}\right)y=0.$$

Thus

$$xP(x)=1, \qquad x^2Q(x)=x^2-\nu^2,$$

which are analytic at the origin. Therefore $x=0$ is a regular singular point.

8.1. Indicial equation

Insert (12) into (17). The derivative terms give

$$x^2y''+xy'-\nu^2y=\sum_{n=0}^{\infty}\left[(n+r)^2-\nu^2\right]a_n x^{n+r},$$

while

$$x^2y=\sum_{n=0}^{\infty}a_n x^{n+r+2}.$$

The lowest power is x^r, with coefficient

$$(r^2-\nu^2)a_0.$$

Since $a_0 \neq 0$, the indicial equation is

$$\boxed{r^2-\nu^2=0, \qquad r=\pm\nu.} \tag{18}$$

8.2. Recurrence relation

Shift the x^2y series by two indices. For $n \geq 2$,

$$\left[(n+r)^2 - \nu^2\right] a_n + a_{n-2} = 0.$$

Hence

$$\boxed{a_n = -\frac{a_{n-2}}{(n+r)^2 - \nu^2}, \qquad n \geq 2.} \tag{19}$$

The coefficient of x^{r+1} gives

$$\left[(r+1)^2 - \nu^2\right] a_1 = 0.$$

For a generic indicial root, this implies $a_1 = 0$, so the nonzero coefficients advance in steps of two.

9. The Bessel Function J_0

For $\nu = 0$, the indicial equation has the repeated root $r = 0$. Choose $a_0 = 1$. The recurrence becomes

$$a_n = -\frac{a_{n-2}}{n^2}.$$

Thus

$$\begin{aligned} a_2 &= -\frac{1}{4}, \\ a_4 &= -\frac{a_2}{16} = \frac{1}{64}, \\ a_6 &= -\frac{a_4}{36} = -\frac{1}{2304}, \end{aligned}$$

and every odd coefficient vanishes. Therefore

$$\boxed{J_0(x) = 1 - \frac{x^2}{4} + \frac{x^4}{64} - \frac{x^6}{2304} + \cdots.} \tag{20}$$

The complete series is

$$\boxed{J_0(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(k!)^2} \left(\frac{x}{2}\right)^{2k}.} \tag{21}$$

The normalization $J_0(0) = 1$ fixes the arbitrary scale.

For general ν, the solution associated with $r = \nu$ is conventionally normalized as

$$\boxed{J_\nu(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(k+\nu+1)} \left(\frac{x}{2}\right)^{2k+\nu}.} \tag{22}$$

For nonnegative integer $\nu = m$, this series is finite at the origin and begins as x^m.

10. The Second Frobenius Solution

Let $r_1 \geq r_2$ be the indicial roots.

- If $r_1 - r_2$ is not an integer, two independent Frobenius series usually result.
- If the roots coincide, the second solution generally contains a logarithm.
- If the roots differ by an integer, the recurrence for the smaller root may fail, and the second solution may contain a logarithm.

For Bessel's equation with noninteger ν, J_ν and $J_{-\nu}$ are independent. For integer order m, however,

$$J_{-m}(x) = (-1)^m J_m(x),$$

so they do not form a basis. The second independent solution is the Bessel function of the second kind $Y_m(x)$, which has a singularity at the origin. For order zero,

$$Y_0(x) \sim \frac{2}{\pi} \ln x \qquad (x \to 0^+),$$

up to an added finite constant and higher-order terms.

Common pitfall

The indicial roots alone do not guarantee two independent pure Frobenius series. When the roots coincide or differ by an integer, the second solution requires special care and may contain $\ln x$.

11. Cylindrical Waves and Space Physics

In cylindrical coordinates (r, ϕ, z), separation of variables in wave, diffusion, and Helmholtz-type equations often produces the radial equation

$$r^2 R'' + rR' + (k_\perp^2 r^2 - m^2)R = 0. \tag{23}$$

With

$$x = k_\perp r,$$

equation (23) becomes Bessel's equation of integer order m:

$$x^2 R_{xx} + xR_x + (x^2 - m^2)R = 0.$$

Thus

$$R(r) = AJ_m(k_\perp r) + BY_m(k_\perp r). \tag{24}$$

If the physical domain includes the axis $r = 0$ and the field must remain finite there, Y_m is excluded and

$$R(r) = AJ_m(k_\perp r).$$

If an outer boundary at $r = a$ requires $R(a) = 0$, then

$$J_m(k_\perp a) = 0. \tag{25}$$

Only discrete values of $k_\perp$ satisfy this condition. Boundary conditions therefore convert a continuous parameter into a discrete set of radial modes.

Applications include waveguide modes, cylindrical plasma columns, magnetic flux tubes, and approximately axisymmetric structures. The precise boundary condition depends on the physical field being modeled; it may involve R, its derivative, or a combination of both.

> Regularity is a physical selection rule. The differential equation permits both J_m and Y_m away from the origin. The requirement that a field remain finite on the axis removes the singular solution only when the axis belongs to the physical domain.

12. Method-Selection Guide

For

$$y'' + P(x)y' + Q(x)y = 0,$$

use the following sequence:

1. Normalize the equation and identify the proposed expansion point.
2. If P and Q are analytic there, use an ordinary power series.
3. If the point is singular, test whether $(x - x_0)P$ and $(x - x_0)^2Q$ are analytic.
4. At a regular singular point, use a Frobenius series and determine the indicial roots from the lowest power.
5. Derive the full recurrence separately for each admissible root.
6. Check whether the roots coincide or differ by an integer before assuming that two pure Frobenius series exist.
7. Apply regularity, boundary, and normalization conditions only after the local solution structure is known.

13. Summary

- A power-series substitution converts a differential equation into a recurrence relation among coefficients.
- At an ordinary point, the two free coefficients correspond to the two initial conditions.
- A regular singular point allows controlled poles in the normalized coefficients and calls for a Frobenius series.

- The indicial equation determines the leading exponent, while higher powers determine the recurrence.
- Bessel functions arise from Frobenius analysis of cylindrical radial equations.
- Physical regularity and boundary conditions select the acceptable solution and may discretize the allowed modes.