GES-207-4 — Mathematical Methods of Physics

National Institute for Space Research (INPE)
Graduate Program in Space Geophysics (PPGGES)

Lecture Notes 14

Variable-Coefficient ODEs and the Euler Equation

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Learning goals and study scope. This self-contained lecture explains why variable coefficients require methods beyond the constant-coefficient characteristic equation. We focus on the Euler–Cauchy equation, whose scale invariance permits power-law solutions. The three root cases, the logarithmic change of variable, nonhomogeneous forcing, resonance, domain restrictions, and an application to radial potential fields are developed in detail. Power series and the Frobenius method begin in Lecture 15.

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1. Why Variable Coefficients Change the Problem

A general linear second-order ordinary differential equation is

$$a(x)y'' + b(x)y' + c(x)y = g(x). \tag{1}$$

On an interval where $a(x) \neq 0$, division by $a(x)$ gives the normalized form

$$\boxed{y'' + P(x)y' + Q(x)y = R(x),} \tag{2}$$

where $P = b/a$, $Q = c/a$, and $R = g/a$.

For constant coefficients, the trial $y = e^{rx}$ works because every derivative is a constant multiple of the same exponential. When P and Q depend on x, substitution of e^{rx} generally leaves uncancelled functions of x. No single characteristic polynomial solves every variable-coefficient equation.

Variable coefficients arise naturally from:

- geometry, especially radial equations in cylindrical or spherical coordinates;
- nonuniform density, temperature, conductivity, or magnetic field;
- coordinate transformations;
- linearization about a spatially varying equilibrium.

1.1. The interval of the equation

The zeros of $a(x)$ and the singularities of P, Q, and R divide the real axis into intervals. Initial data determine a solution on one such interval; the differential equation alone does not justify continuation through a singular point.

For example,

$$x^2y'' + xy' - 4y = 0$$

can be normalized only for $x \neq 0$:

$$y'' + \frac{1}{x}y' - \frac{4}{x^2}y = 0.$$

The intervals $(-\infty, 0)$ and $(0, \infty)$ must be treated separately.

2. Euler–Cauchy Equations

Homogeneous Euler–Cauchy equation

A second-order homogeneous Euler–Cauchy equation has the form

$$\boxed{x^2y'' + \alpha xy' + \beta y = 0, \qquad x \neq 0,} \tag{3}$$

where α and β are constants.

Each derivative is multiplied by the corresponding power of x:

$$x^2y'', \qquad xy', \qquad y.$$

If x is rescaled to λx, all three terms transform with the same overall power. This scale invariance suggests a power law rather than an exponential in x.

Take

$$y = x^m, \tag{4}$$

initially on $x > 0$. Then

$$y' = mx^{m-1}, \qquad y'' = m(m-1)x^{m-2}.$$

Substitution into (3) gives

$$[m(m-1) + \alpha m + \beta]\, x^m = 0.$$

For $x > 0$, $x^m \neq 0$, so the exponent satisfies

$$\boxed{m(m-1) + \alpha m + \beta = 0.} \tag{5}$$

This is the Euler auxiliary equation. Its roots play the same role as the characteristic roots of a constant-coefficient equation.

> The symbol m is an exponent, not a physical mass. It may be real, repeated, or complex. The resulting solution must be interpreted on an interval that does not cross $x = 0$.

3. The Three Root Cases

Write the auxiliary polynomial as

$$p(m) = m(m-1) + \alpha m + \beta = m^2 + (\alpha - 1)m + \beta.$$

3.1. Two distinct real roots

If $m_1 \neq m_2$ are real, then

$$\boxed{y = C_1x^{m_1} + C_2x^{m_2}, \qquad x > 0.} \tag{6}$$

Their Wronskian is

$$W = (m_2 - m_1)x^{m_1+m_2-1},$$

which is nonzero for $x > 0$.

Example 14.1: distinct power-law modes

Solve

$$x^2y'' + xy' - 4y = 0, \qquad x > 0.$$

The auxiliary equation is

$$m(m-1) + m - 4 = m^2 - 4 = 0,$$

so $m = 2$ or $m = -2$. Therefore

$$\boxed{y = C_1x^2 + C_2x^{-2}.} \tag{7}$$

One mode grows with x, while the other decays. Which mode is physically acceptable depends on the domain and boundary conditions.

3.2. A repeated real root

If $p(m)$ has a repeated root m_0, one solution is x^{m_0}. A second independent solution is

$$x^{m_0}\ln x.$$

Thus

$$\boxed{y = x^{m_0}(C_1 + C_2\ln x), \qquad x > 0.} \tag{8}$$

The logarithm is the Euler counterpart of the factor x multiplying e^{rx} for a repeated root in a constant-coefficient equation.

Example 14.2: repeated Euler root

For

$$x^2y'' - 3xy' + 4y = 0,$$

the auxiliary equation is

$$m(m-1) - 3m + 4 = m^2 - 4m + 4 = (m-2)^2.$$

Hence

$$\boxed{y = x^2(C_1 + C_2\ln x).} \tag{9}$$

The Wronskian of x^2 and $x^2\ln x$ is x^3, so the two functions are independent for $x > 0$.

3.3. Complex-conjugate roots

Suppose

$$m = \lambda \pm i\mu, \qquad \mu > 0.$$

Since

$$x^m = e^{m\ln x} = x^\lambda e^{\pm i\mu \ln x},$$

Euler's formula gives the real basis

$$\boxed{y = x^\lambda \left[C_1 \cos(\mu \ln x) + C_2 \sin(\mu \ln x)\right].} \tag{10}$$

The oscillation is periodic in $\ln x$, not in x. Successive zeros are geometrically rather than arithmetically spaced.

Example 14.3: oscillation in logarithmic distance

Solve

$$x^2 y'' + xy' + 4y = 0.$$

The auxiliary equation is $m^2 + 4 = 0$, so $m = \pm 2i$. Therefore

$$\boxed{y = C_1 \cos(2\ln x) + C_2 \sin(2\ln x), \qquad x > 0.}$$

The solution repeats when $\ln x$ increases by π, which means that x is multiplied by e^π.

4. The Logarithmic Change of Variable

The power-law trial is not an isolated trick. It follows from a transformation that converts every Euler equation into a constant-coefficient equation.

On $x > 0$, define

$$t = \ln x, \qquad Y(t) = y(e^t). \tag{11}$$

Since $\mathrm{d}t/\,\mathrm{d}x = 1/x$,

$$y' = \frac{1}{x} Y', \qquad xy' = Y'.$$

Differentiate once more:

$$y'' = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{x}Y'\right) = -\frac{1}{x^2}Y' + \frac{1}{x^2}Y'',$$

so

$$\boxed{x^2 y'' = Y'' - Y'.} \tag{12}$$

Substitution into (3) gives

$$\boxed{Y'' + (\alpha - 1)Y' + \beta Y = 0,} \tag{13}$$

which has constant coefficients.

The exponential trial $Y = e^{mt}$ becomes

$$Y = e^{m\ln x} = x^m,$$

and the characteristic equation of (13) is precisely the auxiliary equation (5). A repeated root produces $te^{m_0 t}$, which becomes $x^{m_0}\ln x$. Complex roots produce trigonometric functions of $t = \ln x$.

4.1. Intervals with $x < 0$

On $x < 0$, use $t = \ln|x|$. The identities

$$xy' = Y', \qquad x^2y'' = Y'' - Y'$$

remain valid. Real solutions can therefore be written with $|x|$:

$$|x|^m, \quad |x|^{m_0}\ln|x|, \quad |x|^\lambda\cos(\mu\ln|x|), \quad |x|^\lambda\sin(\mu\ln|x|).$$

The constants on $x < 0$ need not match those on $x > 0$ because the differential equation is singular at the origin.

5. Nonhomogeneous Euler Equations

A nonhomogeneous Euler equation has the form

$$x^2y'' + \alpha xy' + \beta y = g(x). \tag{14}$$

The logarithmic transformation turns it into

$$Y'' + (\alpha - 1)Y' + \beta Y = g(e^t),$$

a constant-coefficient equation in t.

5.1. Power-law forcing

If $g(x) = Kx^s$ and $p(s) \neq 0$, try $y_p = Ax^s$. Since

$$L[x^s] = p(s)x^s,$$

we obtain

$$\boxed{y_p = \frac{K}{p(s)}x^s, \qquad p(s) \neq 0.} \tag{15}$$

Example 14.4: nonresonant forcing

Solve

$$x^2y'' - xy' - 3y = x^2, \qquad x > 0.$$

The auxiliary polynomial is

$$p(m) = m(m-1) - m - 3 = (m-3)(m+1),$$

so

$$y_h = C_1 x^3 + C_2 x^{-1}.$$

For the forcing exponent $s = 2$,

$$p(2) = (2-3)(2+1) = -3 \neq 0.$$

Thus

$$y_p = -\frac{1}{3}x^2,$$

and

$$\boxed{y = C_1 x^3 + C_2 x^{-1} - \frac{1}{3}x^2.} \tag{16}$$

5.2. Resonant power-law forcing

If s is a root of $p(m)$, then x^s already belongs to the homogeneous solution. The trial must be multiplied by $\ln x$. If the root has multiplicity k, multiply by $(\ln x)^k$:

$$\boxed{y_p = Ax^s(\ln x)^k.} \tag{17}$$

This is the transformed version of multiplying a resonant exponential trial by t^k in a constant-coefficient equation.

Example 14.5: forcing at a repeated Euler root

Consider

$$x^2y'' - 3xy' + 4y = x^2.$$

The auxiliary polynomial is $(m-2)^2$, so the forcing exponent $s = 2$ is a root of multiplicity two. Take

$$y_p = Ax^2(\ln x)^2.$$

The transformation $t = \ln x$ converts the equation into

$$(D-2)^2 Y = e^{2t}, \qquad D = \frac{\mathrm{d}}{\mathrm{d}t}.$$

For $Y_p = At^2e^{2t}$,

$$(D-2)^2 Y_p = 2Ae^{2t},$$

so $A = 1/2$. Therefore

$$\boxed{y = x^2(C_1 + C_2 \ln x) + \frac{1}{2}x^2(\ln x)^2.} \tag{18}$$

Common pitfall

The word resonance here means that the forcing duplicates a homogeneous mode after the logarithmic change of variable. It does not necessarily imply a periodic physical oscillation. The algebraic signature is $p(s) = 0$.

6. Initial Conditions

Initial conditions must be imposed at a point $x_0 \neq 0$. For example, solve

$$x^2y'' + xy' - 4y = 0, \qquad y(1) = 3, \qquad y'(1) = 0.$$

From Example 14.1,

$$y = C_1x^2 + C_2x^{-2}, \qquad y' = 2C_1x - 2C_2x^{-3}.$$

At $x = 1$,

$$C_1 + C_2 = 3, \qquad 2C_1 - 2C_2 = 0.$$

Thus $C_1 = C_2 = 3/2$ and

$$\boxed{y(x) = \frac{3}{2}\left(x^2 + x^{-2}\right), \qquad x > 0.} \tag{19}$$

The maximal interval containing the initial point is $(0, \infty)$.

7. Physical Application: Radial Potential Fields

In a source-free region, a scalar potential satisfying Laplace's equation can be separated in spherical coordinates. The radial factor for angular degree ℓ satisfies

$$r^2R'' + 2rR' - \ell(\ell+1)R = 0, \qquad r > 0. \tag{20}$$

This is an Euler equation. The auxiliary equation is

$$\begin{aligned} m(m-1) + 2m - \ell(\ell+1) &= 0, \\ m^2 + m - \ell(\ell+1) &= 0, \\ (m-\ell)(m+\ell+1) &= 0. \end{aligned}$$

Hence

$$\boxed{R(r) = Ar^{\ell} + Br^{-(\ell+1)}.} \tag{21}$$

Inside a region containing the origin, regularity normally excludes $r^{-(\ell+1)}$. In an exterior region extending to infinity, decay normally excludes r^{ℓ}. The radial dependence is therefore selected by the physical domain and boundary conditions.

This structure appears in gravitational, electrostatic, and magnetic scalar potentials.

In space physics, spherical-harmonic descriptions of planetary magnetic fields use precisely these radial power laws. Internal and external sources produce different powers of r.

No single nonzero solution of (20) is both regular at $r = 0$ and decaying as $r \to \infty$ across one source-free domain. A physical source or boundary separates interior and exterior regions, and matching conditions connect their solutions.

8. Method Checklist and Common Errors

For an Euler–Cauchy equation:

1. State whether the work is on $x > 0$ or $x < 0$.
2. Move the equation into the form $x^2y'' + \alpha xy' + \beta y = g(x)$.
3. Form $p(m) = m(m-1) + \alpha m + \beta$.
4. Choose the correct homogeneous basis for distinct, repeated, or complex roots.
5. For power-law forcing, test whether $p(s)$ vanishes before choosing the particular solution.
6. Apply initial or boundary conditions only after the complete solution is written.
7. Substitute the result into the original equation and state its interval of validity.

Common pitfall

The coefficient of m in the auxiliary polynomial is $\alpha - 1$, not α. The term $m(m-1)$ comes from the second derivative. Omitting the $-m$ is one of the most common algebraic errors in Euler equations.

9. Summary

- Variable coefficients usually prevent the direct use of an exponential trial in the original variable.
- Euler–Cauchy equations are scale invariant and admit power-law modes.
- The substitution $t = \ln|x|$ converts an Euler equation into a constant-coefficient equation.
- Repeated roots introduce $\ln|x|$, while complex roots produce oscillations in logarithmic distance.
- A power-law forcing resonates when its exponent is a root of the auxiliary polynomial.
- Radial potential equations provide a direct physical application of Euler power laws.