GES-207-4 — Mathematical Methods of Physics

National Institute for Space Research (INPE)
Graduate Program in Space Geophysics (PPGGES)

Lecture Notes 13

Second-Order Linear ODEs with Constant Coefficients

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Learning goals and study scope. This self-contained lecture develops second-order linear differential equations with constant coefficients from their algebraic structure to their physical interpretation. We derive the characteristic equation, explain all three root cases, and separate homogeneous and forced responses. The damped and driven oscillator provides the main physical model, including transient behavior, phase lag, and resonance. We also connect the same mathematics to cyclotron motion and Green-function representations.

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1. Structure of a Second-Order Linear Equation

A general linear second-order equation is

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = g(x), \tag{1}$$

where the coefficient of y'' is nonzero on the interval of interest. This lecture focuses on constant coefficients:

$$\boxed{ay'' + by' + cy = g(x), \qquad a \neq 0.} \tag{2}$$

Two initial conditions are normally required, for example

$$y(x_0) = y_0, \qquad y'(x_0) = v_0.$$

They determine the two constants in the general solution.

Homogeneous and particular parts

The associated homogeneous equation is

$$ay'' + by' + cy = 0. \tag{3}$$

If y_h is the general homogeneous solution and y_p is any one solution of the full equation (2), then

$$\boxed{y = y_h + y_p} \tag{4}$$

is the general solution of the nonhomogeneous equation.

Indeed, linearity gives $L[y_h + y_p] = L[y_h] + L[y_p] = 0 + g = g$. If $y_{p,1}$ and $y_{p,2}$ are two particular solutions, their difference satisfies the homogeneous equation. Thus different choices of y_p lead to the same full family after the homogeneous constants are included.

2. Superposition and Linear Independence

If y_1 and y_2 solve (3), then every linear combination

$$y_h = C_1y_1 + C_2y_2$$

also solves it. To span the full two-dimensional solution space, y_1 and y_2 must be linearly independent.

Wronskian

The Wronskian of two differentiable functions is

$$\boxed{W[y_1, y_2](x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_1' y_2.} \tag{5}$$

For solutions of the same linear homogeneous second-order equation with continuous coefficients, a nonzero Wronskian at one point guarantees linear independence throughout the interval.

After division of (1) by $a_2(x)$, write the homogeneous equation as

$$y'' + P(x)y' + Q(x)y = 0.$$

Direct differentiation of (5) gives Abel's identity

$$W'(x) = -P(x)W(x), \qquad W(x) = W(x_0)e^{-\int_{x_0}^{x} P(s)\,\mathrm{d}s}. \tag{6}$$

Consequently, the Wronskian of two solutions is either identically zero or never zero on the interval.

3. The Characteristic Equation

For constant coefficients, exponential functions are natural because every derivative reproduces the same function. Try

$$y = e^{rx}.$$

Then $y' = re^{rx}$ and $y'' = r^2 e^{rx}$. Substitution into (3) yields

$$(ar^2 + br + c)e^{rx} = 0.$$

Since $e^{rx} \neq 0$,

$$\boxed{ar^2 + br + c = 0.} \tag{7}$$

This is the characteristic equation. Its discriminant

$$\Delta = b^2 - 4ac$$

determines the form of the homogeneous solution.

3.1. Two distinct real roots

If $\Delta > 0$, the roots r_1 and r_2 are real and distinct. Then

$$\boxed{y_h = C_1 e^{r_1 x} + C_2 e^{r_2 x}.} \tag{8}$$

Their Wronskian is

$$W = (r_2 - r_1)e^{(r_1+r_2)x},$$

which never vanishes because $r_1 \neq r_2$.

Example 13.1: distinct real roots and initial data

Solve

$$y'' - 5y' + 6y = 0, \qquad y(0) = 1, \qquad y'(0) = 0.$$

The characteristic equation factors as

$$r^2 - 5r + 6 = (r - 2)(r - 3) = 0.$$

Therefore

$$y = C_1e^{2x} + C_2e^{3x}.$$

At $x = 0$,

$$C_1 + C_2 = 1, \qquad 2C_1 + 3C_2 = 0.$$

Solving gives $C_1 = 3$ and $C_2 = -2$, so

$$\boxed{y(x) = 3e^{2x} - 2e^{3x}.}$$

Both initial conditions and the differential equation should be checked after the constants are found.

3.2. A repeated real root

If $\Delta = 0$, the characteristic equation has one repeated root $r = -b/(2a)$. The exponential e^{rx} provides only one solution. A second independent solution is obtained by reduction of order or by taking a limit of the distinct-root case:

$$\boxed{y_h = (C_1 + C_2x)e^{rx}.} \tag{9}$$

The Wronskian of e^{rx} and xe^{rx} is e^{2rx}, so the functions are independent.

Example 13.2: repeated root

For

$$y'' - 6y' + 9y = 0,$$

the characteristic polynomial is $(r - 3)^2$. Hence

$$\boxed{y = (C_1 + C_2x)e^{3x}.}$$

The factor x is essential. Writing two copies of e^{3x} would still give only one independent solution.

3.3. Complex-conjugate roots

If $\Delta < 0$, write

$$r = \alpha \pm i\beta, \qquad \beta > 0.$$

Euler's formula gives

$$e^{(\alpha+i\beta)x} = e^{\alpha x}(\cos\beta x + i\sin\beta x).$$

A real basis is therefore

$$\boxed{y_h = e^{\alpha x}\left(C_1\cos\beta x + C_2\sin\beta x\right).} \tag{10}$$

The real part α controls exponential growth or decay. The imaginary part β controls oscillation.

Example 13.3: decaying oscillation

Solve

$$y'' + 2y' + 5y = 0.$$

The roots are

$$r = -1 \pm 2i,$$

so

$$\boxed{y = e^{-x}(C_1\cos 2x + C_2\sin 2x).}$$

The envelope decays as e^{-x} and the angular frequency is 2.

4. The Damped Harmonic Oscillator

A mass attached to a linear spring and a viscous damper satisfies

$$m\ddot{q} + b\dot{q} + kq = 0.$$

After division by m, define

$$2\gamma = \frac{b}{m}, \qquad \omega_0^2 = \frac{k}{m},$$

to obtain

$$\boxed{\ddot{q} + 2\gamma\dot{q} + \omega_0^2 q = 0.} \tag{11}$$

The characteristic roots are

$$r = -\gamma \pm \sqrt{\gamma^2 - \omega_0^2}. \tag{12}$$

4.1. Underdamped motion

If $0 \le \gamma < \omega_0$, define

$$\omega_d = \sqrt{\omega_0^2 - \gamma^2}.$$

Then

$$\boxed{q(t) = e^{-\gamma t}\left[A\cos(\omega_d t) + B\sin(\omega_d t)\right].} \tag{13}$$

The motion oscillates with damped frequency ω_d, while its amplitude decays on the timescale $1/\gamma$. Damping lowers the oscillation frequency relative to the undamped value ω_0.

4.2. Critical and overdamped motion

If $\gamma = \omega_0$, the root $r = -\gamma$ is repeated:

$$\boxed{q(t) = (A + Bt)e^{-\gamma t}.} \tag{14}$$

This is critical damping, the boundary between oscillatory and nonoscillatory decay.

If $\gamma > \omega_0$, both roots are real and negative:

$$q(t) = Ae^{r_1 t} + Be^{r_2 t}. \tag{15}$$

The motion decays without oscillating. One exponential usually dominates at late times because its root lies closer to zero.

> Critical damping gives the fastest return to equilibrium without oscillation for the ideal linear model. It does not mean the amplitude becomes zero after a finite time.

5. Periodic Forcing and the Steady Response

Now consider

$$\boxed{\ddot{q} + 2\gamma\dot{q} + \omega_0^2 q = F_0 \cos(\Omega t).} \tag{16}$$

The complete solution is

$$q(t) = q_{\text{transient}}(t) + q_{\text{steady}}(t). \tag{17}$$

The transient solves the homogeneous equation and carries the initial conditions. When $\gamma > 0$, it decays. The steady solution oscillates at the driving frequency Ω, not at the natural frequency.

Take

$$q_{\text{steady}} = a\cos\Omega t + b\sin\Omega t.$$

Substitution into (16) gives

$$\begin{aligned}(\omega_0^2 - \Omega^2)a + 2\gamma\Omega b &= F_0,\\ (\omega_0^2 - \Omega^2)b - 2\gamma\Omega a &= 0.\end{aligned}$$

Solving,

$$a = \frac{F_0(\omega_0^2 - \Omega^2)}{(\omega_0^2 - \Omega^2)^2 + 4\gamma^2\Omega^2}, \tag{18}$$

$$b = \frac{2\gamma\Omega F_0}{(\omega_0^2 - \Omega^2)^2 + 4\gamma^2\Omega^2}. \tag{19}$$

Write the result as

$$q_{\text{steady}}(t) = A(\Omega)\cos(\Omega t - \delta). \tag{20}$$

Since $a = A\cos\delta$ and $b = A\sin\delta$,

$$\boxed{A(\Omega) = \frac{F_0}{\sqrt{(\omega_0^2 - \Omega^2)^2 + 4\gamma^2\Omega^2}},} \tag{21}$$

and

$$\boxed{\tan\delta = \frac{2\gamma\Omega}{\omega_0^2 - \Omega^2}.} \tag{22}$$

The correct quadrant for δ must be chosen from the signs of numerator and denominator, preferably with a two-argument arctangent.

Example 13.4: a complete forced response

Find a particular periodic solution of

$$y'' + 2y' + 5y = 4\cos 2x.$$

Take $y_p = a\cos 2x + b\sin 2x$. Substitution gives

$$a + 4b = 4, \qquad b - 4a = 0.$$

Thus $a = 4/17$ and $b = 16/17$, so

$$\boxed{y_p = \frac{4}{17}\cos 2x + \frac{16}{17}\sin 2x = \frac{4}{\sqrt{17}}\cos(2x - \arctan 4).} \tag{23}$$

The general solution also contains the transient $e^{-x}(C_1\cos 2x + C_2\sin 2x)$.

6. Resonance

The displacement amplitude (21) becomes large when its denominator is small. For weak damping, the peak lies near $\Omega = \omega_0$. Differentiating the squared denominator shows that the nonzero-frequency peak occurs at

$$\boxed{\Omega_{\text{r}} = \sqrt{\omega_0^2 - 2\gamma^2},} \tag{24}$$

provided $\gamma < \omega_0/\sqrt{2}$. Increasing damping lowers and broadens the displacement-resonance peak.

The frequency that maximizes displacement, velocity, absorbed power, or energy need not be identical when damping is finite. The statement "resonance occurs at ω_0" is an accurate weak-damping approximation but not a universal exact formula.

6.1. Exact resonance without damping

If $\gamma = 0$ and $\Omega = \omega_0$, the equation becomes

$$\ddot{q} + \omega_0^2 q = F_0 \cos(\omega_0 t). \tag{25}$$

A trial proportional to $\cos \omega_0 t$ fails because it is already a homogeneous solution. Multiply the trial by t and take

$$q_p = Ct \sin \omega_0 t.$$

Then

$$\ddot{q}_p + \omega_0^2 q_p = 2C\omega_0 \cos \omega_0 t,$$

so

$$\boxed{q_p(t) = \frac{F_0}{2\omega_0} t \sin \omega_0 t.} \tag{26}$$

The linearly growing envelope is called a secular response. In a physical system, damping, nonlinearities, finite driving time, or energy depletion eventually limits the growth.

7. Cyclotron Motion

For a particle of charge q and mass m in a uniform magnetic field

$$\boldsymbol{B} = B_0 \hat{\boldsymbol{z}},$$

the Lorentz equation without an electric field is

$$m\dot{\boldsymbol{v}} = q\boldsymbol{v} \times \boldsymbol{B}.$$

Its perpendicular components satisfy

$$\begin{aligned} \dot{v}_x &= \omega_c v_y, \\ \dot{v}_y &= -\omega_c v_x, \end{aligned}$$

where $\omega_c = qB_0/m$ is the signed gyrofrequency. Differentiate once more:

$$\begin{aligned} \ddot{v}_x + \Omega_c^2 v_x &= 0, \\ \ddot{v}_y + \Omega_c^2 v_y &= 0, \end{aligned}$$

with

$$\boxed{\Omega_c = \frac{|q|B_0}{m}.} \tag{27}$$

Each velocity component is a harmonic oscillator. The two components are a quarter period out of phase, producing circular motion in velocity space and, after integration, in the plane perpendicular to $\boldsymbol{B}$. The sign of q sets the direction of rotation but not the positive frequency Ω_c.

The magnetic force does no work because

$$\boldsymbol{v} \cdot (q\boldsymbol{v} \times \boldsymbol{B}) = 0.$$

Thus the speed and kinetic energy remain constant even though the direction of the velocity changes continuously.

8. Green-Function Form

For the undamped forced problem with zero initial data,

$$\ddot{q} + \omega_0^2 q = f(t), \qquad q(0) = \dot{q}(0) = 0, \tag{28}$$

the solution can be written as

$$\boxed{q(t) = \frac{1}{\omega_0} \int_0^t \sin[\omega_0(t-s)]\, f(s)\, \mathrm{d}s.} \tag{29}$$

The kernel

$$G(t-s) = \frac{\sin[\omega_0(t-s)]}{\omega_0}$$

is the causal impulse response. A force applied at time s produces a delayed oscillatory contribution at time t.

To verify (29), differentiate using the Leibniz rule:

$$\begin{aligned} \dot{q}(t) &= \int_0^t \cos[\omega_0(t-s)] f(s)\, \mathrm{d}s, \\ \ddot{q}(t) &= f(t) - \omega_0 \int_0^t \sin[\omega_0(t-s)] f(s)\, \mathrm{d}s \\ &= f(t) - \omega_0^2 q(t). \end{aligned}$$

The integral also gives $q(0) = \dot{q}(0) = 0$ directly.

For the underdamped operator, the corresponding causal kernel is

$$G(t-s) = \frac{e^{-\gamma(t-s)} \sin[\omega_d(t-s)]}{\omega_d}, \qquad t \geq s. \tag{30}$$

The exponential suppresses the memory of forcing in the distant past.

9. Method Checklist and Common Errors

For a constant-coefficient second-order problem:

1. Solve the homogeneous characteristic equation and identify the correct root case.
2. Construct two linearly independent homogeneous solutions.
3. Find one particular solution using a form suited to the forcing.

4. If the trial duplicates a homogeneous term, multiply it by a sufficient power of the independent variable.

5. Combine $y_h + y_p$ before applying initial conditions.

6. Check the final expression in the original equation.

Common pitfall

For complex roots $\alpha \pm i\beta$, the real solution contains the common factor $e^{\alpha x}$. Omitting it changes decaying or growing oscillation into an undamped one. For a repeated root, the second solution is xe^{rx}, not a second copy of e^{rx}.

10. Summary

- The characteristic roots determine the natural homogeneous modes.
- Distinct, repeated, and complex roots require different real bases.
- The Wronskian tests whether two solutions span the homogeneous solution space.
- A forced solution separates into a transient fixed by initial conditions and a steady response fixed by the forcing.
- Damping controls decay, shifts and broadens the displacement-resonance peak, and prevents unlimited resonant growth.
- Harmonic-oscillator equations describe mechanical oscillations, cyclotron motion, and impulse responses across many physical systems.