GES-207-4 — Mathematical Methods of Physics

National Institute for Space Research (INPE)
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Lecture Notes 12

First-Order ODEs: Integrating Factor, Bernoulli, and Riccati Equations

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Learning goals and study scope. This self-contained lecture develops three important classes of first-order ordinary differential equations. We derive the integrating-factor method for linear equations, interpret its solution as a weighted memory of past forcing, and apply it to a source-and-loss model. We then show how Bernoulli and Riccati equations can be transformed into linear equations when their special nonlinear structure is recognized. Domain restrictions, solutions lost during division, and finite-time singularities are treated as part of the solution rather than as afterthoughts.

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1. First-Order ODEs and Initial-Value Problems

An ordinary differential equation relates an unknown function to its derivatives with respect to one independent variable. An equation has order n if $y^{(n)}$ is its highest derivative. A general first-order equation may be written as

$$F(x, y, y') = 0 \tag{1}$$

or, when it can be solved for the derivative,

$$y' = f(x, y). \tag{2}$$

An initial-value problem supplements the differential equation with a value

$$y(x_0) = y_0.$$

The equation determines allowable trajectories; the initial condition selects one trajectory from the family.

Linear first-order equation

A first-order equation is linear in y if it can be written on an interval as

$$\boxed{y' + P(x)y = Q(x).} \tag{3}$$

The coefficient functions P and Q may depend on x, but y and y' occur only to the first power and are not multiplied together. The equation is homogeneous when $Q = 0$ and nonhomogeneous when $Q \neq 0$.

For example, $y' + xy = \sin x$ is linear, whereas $y' + xy^2 = 0$ and $yy' + y = x$ are nonlinear. The appearance of a nonlinear power does not imply that no systematic method exists; Bernoulli and Riccati equations are two important structured exceptions.

2. The Integrating-Factor Method

We seek a function $\mu(x)$ that converts the left side of (3) into the derivative of a product. Multiply the equation by μ:

$$\mu y' + \mu P y = \mu Q.$$

The product rule gives

$$(\mu y)' = \mu y' + \mu' y.$$

The two expressions agree if

$$\mu' = P\mu. \tag{4}$$

Integrating $\mu'/\mu = P$ yields

$$\boxed{\mu(x) = e^{\int P(x)\,\mathrm{d}x}.} \tag{5}$$

Any nonzero multiplicative constant in μ cancels from the final solution, so it is normally set equal to one.

With this choice,

$$(\mu y)' = \mu Q.$$

Integration gives

$$\boxed{y(x) = \frac{1}{\mu(x)}\left[\int \mu(x)Q(x)\,\mathrm{d}x + C\right].} \tag{6}$$

2.1. Initial-value form

For $y(x_0) = y_0$, integrate from x_0 to x:

$$\mu(x)y(x) - \mu(x_0)y_0 = \int_{x_0}^{x} \mu(s)Q(s)\,\mathrm{d}s.$$

Thus

$$\boxed{y(x) = \frac{\mu(x_0)y_0}{\mu(x)} + \frac{1}{\mu(x)}\int_{x_0}^{x} \mu(s)Q(s)\,\mathrm{d}s.} \tag{7}$$

This form avoids introducing an indefinite-integral constant and separates the response to the initial condition from the response to the forcing.

> The symbol used inside the definite integral is a dummy variable. Writing $\int_{x_0}^{x} \mu(x)Q(x)\,\mathrm{d}x$ obscures the fact that the upper limit x is not the integration variable. The notation s in (7) keeps these roles distinct.

3. A Complete Linear Example

Example 12.1: a singular coefficient

Solve

$$y' + \frac{2}{x}y = x^2, \qquad x > 0, \qquad y(1) = 0. \tag{8}$$

Here $P(x) = 2/x$, so

$$\mu(x) = e^{\int 2/x\,\mathrm{d}x} = e^{2\ln x} = x^2,$$

where $x > 0$ justifies $\ln x$. Multiplication by x^2 gives

$$x^2y' + 2xy = x^4,$$

or

$$(x^2y)' = x^4.$$

Therefore

$$x^2y = \frac{x^5}{5} + C, \qquad y = \frac{x^3}{5} + \frac{C}{x^2}.$$

The initial condition gives $0 = 1/5 + C$, hence $C = -1/5$ and

$$\boxed{y(x) = \frac{x^5 - 1}{5x^2}, \qquad x > 0.} \tag{9}$$

Direct differentiation verifies both the equation and the initial condition.

3.1. Why the domain belongs to the answer

The coefficient $2/x$ is singular at $x = 0$. The equation is continuous on $(-\infty, 0)$ and $(0, \infty)$ separately, but not across the origin. An initial condition at $x = 1$ determines a solution only on an interval contained in $(0, \infty)$. Formula (9) also diverges as $x \to 0^+$.

If instead one worked on $x < 0$, then

$$e^{\int 2/x\,\mathrm{d}x} = e^{2\ln|x|} = x^2.$$

The same algebraic integrating factor appears, but the interval still cannot cross zero. A solution consists of a formula together with an interval on which the original equation and all transformations are valid.

Common pitfall

Do not simplify $e^{\int 2/x\,\mathrm{d}x}$ as $e^{2\ln x}$ without first specifying $x > 0$. On a general interval avoiding zero, the antiderivative is $2\ln|x|$. The final factor x^2 is valid on either side, but the reasoning must respect the domain.

4. Source, Loss, and Memory

Many physical balance laws have the form

$$\frac{\mathrm{d}N}{\mathrm{d}t} = S(t) - \frac{N}{\tau}, \qquad \tau > 0. \tag{10}$$

Here $N(t)$ may represent a particle population, number density, energy content, or another accumulated quantity. The function $S(t)$ is a source, while N/τ models linear loss with characteristic time τ.

In standard form,

$$\frac{\mathrm{d}N}{\mathrm{d}t} + \frac{1}{\tau}N = S(t),$$

so the integrating factor is $\mu(t) = e^{t/\tau}$. With $N(0) = N_0$,

$$e^{t/\tau}N(t) - N_0 = \int_0^t e^{s/\tau}S(s)\,\mathrm{d}s,$$

and therefore

$$\boxed{N(t) = N_0 e^{-t/\tau} + \int_0^t e^{-(t-s)/\tau}S(s)\,\mathrm{d}s.} \tag{11}$$

The first term is the surviving fraction of the initial population. A source contribution created at time s is weighted at time t by $e^{-(t-s)/\tau}$. Recent forcing has a larger weight than forcing in the distant past. The solution therefore represents an exponentially weighted memory.

Example 12.2: constant source

If $S(t) = S_0$, then

$$\begin{aligned} N(t) &= N_0 e^{-t/\tau} + S_0 \int_0^t e^{-(t-s)/\tau}\,\mathrm{d}s \\ &= N_0 e^{-t/\tau} + S_0\tau(1 - e^{-t/\tau}). \end{aligned}$$

Thus

$$\boxed{N(t) = S_0\tau + (N_0 - S_0\tau)e^{-t/\tau}.}$$

The equilibrium $N_{\mathrm{eq}} = S_0\tau$ follows directly by setting $\mathrm{d}N/\,\mathrm{d}t = 0$. The solution relaxes exponentially toward that equilibrium.

4.1. Exponentially decaying injection

Consider

$$\frac{\mathrm{d}N}{\mathrm{d}t} = S_0 e^{-t/\tau_s} - \frac{N}{\tau_\ell}, \qquad N(0) = N_0, \tag{12}$$

where τ_s is the source-decay time and τ_ℓ is the loss time. From (11),

$$N(t) = N_0 e^{-t/\tau_\ell} + S_0 e^{-t/\tau_\ell} \int_0^t e^{s(1/\tau_\ell - 1/\tau_s)}\,\mathrm{d}s.$$

If $\tau_s \neq \tau_\ell$, then

$$\boxed{N(t) = N_0 e^{-t/\tau_\ell} + S_0 \frac{e^{-t/\tau_s} - e^{-t/\tau_\ell}}{1/\tau_\ell - 1/\tau_s}.} \tag{13}$$

If $\tau_s = \tau_\ell = \tau$, the exponent inside the integral vanishes and the apparently singular quotient in (13) must be replaced by its limit:

$$\boxed{N(t) = (N_0 + S_0 t)e^{-t/\tau}.} \tag{14}$$

The factor t appears because the source and homogeneous response have the same decay rate, an analogue of resonance in linear differential equations.

5. Bernoulli Equations

Bernoulli equation

A Bernoulli equation has the form

$$\boxed{y' + P(x)y = Q(x)y^n, \qquad n \neq 0, 1.} \tag{15}$$

It is nonlinear in y, but its power-law structure permits a substitution that produces a linear equation.

On an interval where $y \neq 0$, divide (15) by y^n:

$$y^{-n}y' + P(x)y^{1-n} = Q(x).$$

Set

$$v = y^{1-n}. \tag{16}$$

Then

$$v' = (1-n)y^{-n}y'.$$

Multiplication of the divided equation by $1-n$ gives

$$\boxed{v' + (1-n)P(x)v = (1-n)Q(x),} \tag{17}$$

which is linear in v.

The division by y^n assumes $y \neq 0$. If the original equation admits the solution $y = 0$, it must be recorded separately because the substitution $v = y^{1-n}$ may not represent it.

Example 12.3: a Bernoulli equation

Solve

$$y' - \frac{2}{x}y = x^3y^2, \qquad x > 0. \tag{18}$$

Here $n = 2$ and $v = y^{-1}$. Formula (17) gives

$$v' + \frac{2}{x}v = -x^3.$$

This is linear. Its integrating factor is x^2, so

$$(x^2v)' = -x^5.$$

After integration,

$$x^2v = -\frac{x^6}{6} + C, \qquad v = \frac{C}{x^2} - \frac{x^4}{6}.$$

Since $y = 1/v$,

$$\boxed{y(x) = \frac{x^2}{C - x^6/6} = \frac{6x^2}{C_1 - x^6}.} \tag{19}$$

The constants C and $C_1 = 6C$ describe the same one-parameter family. The zero solution $y = 0$ must be added separately because it was lost when the equation was divided by y^2.

For a fixed constant, the interval of validity must exclude both $x = 0$ and any point at which the denominator in (19) vanishes.

5.1. A saturation model

The logistic equation

$$\frac{\mathrm{d}N}{\mathrm{d}t} = rN - \kappa N^2, \qquad r, \kappa > 0, \tag{20}$$

is Bernoulli with $n = 2$:

$$N' - rN = -\kappa N^2.$$

Taking $v = N^{-1}$ gives

$$v' + rv = \kappa.$$

For $N(0) = N_0 > 0$,

$$\boxed{N(t) = \frac{K}{1 + (K/N_0 - 1)e^{-rt}}, \qquad K = \frac{r}{\kappa}.} \tag{21}$$

The linear growth term dominates at low density, while the quadratic loss limits the population to the carrying level K. In a space-physics model, such a term may represent a phenomenological saturation or recombination mechanism. The interpretation and units of r and κ must be specified for the particular system.

6. Riccati Equations

Riccati equation

A Riccati equation has the form

$$\boxed{y' = a(x)y^2 + b(x)y + c(x).} \tag{22}$$

Unlike the Bernoulli equation, it generally contains a nonzero term independent of y. There is no universal elementary solution formula. However, one known particular solution reduces the problem to a linear equation.

Let y_p satisfy

$$y_p' = ay_p^2 + by_p + c.$$

Set

$$y = y_p + \frac{1}{u}. \tag{23}$$

Then

$$y' = y_p' - \frac{u'}{u^2}.$$

Substitution into (22) gives

$$y_p' - \frac{u'}{u^2} = a\left(y_p + \frac{1}{u}\right)^2 + b\left(y_p + \frac{1}{u}\right) + c.$$

Cancel the equation satisfied by y_p and multiply by u^2:

$$-u' = (2ay_p + b)u + a.$$

Therefore

$$\boxed{u' + (2ay_p + b)u = -a,} \tag{24}$$

a first-order linear equation.

Example 12.4: Riccati reduction from a known solution

Consider

$$y' = y^2 - \frac{2}{x^2}, \qquad x > 0. \tag{25}$$

First verify that $y_p = 1/x$ is a particular solution:

$$y_p' = -\frac{1}{x^2}, \qquad y_p^2 - \frac{2}{x^2} = \frac{1}{x^2} - \frac{2}{x^2} = -\frac{1}{x^2}.$$

Here $a = 1$ and $b = 0$. Substitution $y = 1/x + 1/u$ yields

$$u' + \frac{2}{x}u = -1.$$

The integrating factor is x^2, hence

$$(x^2 u)' = -x^2, \qquad u = -\frac{x}{3} + \frac{C}{x^2}.$$

Thus

$$\boxed{y(x) = \frac{1}{x} + \frac{1}{-x/3 + C/x^2} = \frac{1}{x} + \frac{3x^2}{3C - x^3}.} \tag{26}$$

The known solution $y_p = 1/x$ remains part of the complete solution set. It is recovered from the transformed family only as a limiting case, not for a finite value of C.

6.1. Finite-time singularities

The denominator $3C - x^3$ in (26) may vanish. This produces a vertical blow-up even though the coefficient $-2/x^2$ is regular at that point when the point is positive and nonzero. Such behavior is possible for nonlinear equations: the solution may cease to exist at a finite point because the dependent variable becomes unbounded.

This is different from the singular coefficient at $x = 0$. A coefficient singularity belongs to the differential equation, whereas a movable singularity depends on the integration constant and therefore on the initial condition.

7. Choosing the Method

For a first-order equation, proceed in the following order:

1. State an interval on which all coefficients are defined.
2. Rewrite the equation in a recognizable standard form.
3. If it is linear, use an integrating factor.
4. If it is Bernoulli, check for the zero solution and use $v = y^{1-n}$ on intervals where $y \neq 0$.
5. If it is Riccati, look for a supplied or easily guessed particular solution before using $y = y_p + 1/u$.
6. Apply the initial condition and then recheck the interval of validity of the resulting formula.
7. Substitute the final expression into the original equation.

Common pitfall

An integrating factor applies only after the equation has been placed in the form $y' + P(x)y = Q(x)$. If the coefficient of y' is not one, divide by it first and record every point where that division is invalid.

8. Summary

- The integrating factor $\mu = e^{\int P\,\mathrm{d}x}$ turns a linear first-order equation into the product derivative $(\mu y)' = \mu Q$.
- The initial-value formula separates the surviving initial state from the accumulated response to forcing.
- A source-and-loss model produces an exponentially weighted memory integral.
- The Bernoulli substitution $v = y^{1-n}$ linearizes a special power-law nonlinearity, but division may remove the zero solution.

- A Riccati equation becomes linear after $y = y_p + 1/u$ when one particular solution is known.
- Domains, coefficient singularities, and solution blow-ups are part of the mathematical answer.