GES-207-4 — Mathematical Methods of Physics

National Institute for Space Research (INPE)
Graduate Program in Space Geophysics (PPGGES)

Lecture Notes 11

Residues, Poles on the Contour, and Riemann Surfaces

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Learning goals and study scope. This self-contained lecture begins where Lecture 10 ended: the residue is the coefficient of $(z - z_0)^{-1}$ in a Laurent series. We derive why that single coefficient controls a closed contour integral, establish practical residue formulas and the residue theorem, and use them to evaluate real and Fourier integrals. We then study the extra care required when a pole lies on the contour. The final sections introduce branch points, branch cuts, and Riemann surfaces, with logarithms, square roots, and a keyhole contour as concrete examples.

Contents

1 From Laurent Series to Residues **2**
1.1 Why only the coefficient a_{-1} contributes 2

2 Practical Calculation of Residues **3**
2.1 Simple poles 3
2.2 Poles of higher order 3

3 The Residue Theorem **4**
3.1 Proof idea 5

4 Real Integrals from Semicircular Contours **5**
4.1 A rational integral 6
4.2 The exponential determines the half-plane 6

5 Poles on the Contour and Principal Values **7**
5.1 The contribution from a small indentation 7
5.2 The Sokhotski–Plemelj prescription 8
5.3 Causality and resonant particles 8

6 Branch Points and Branch Cuts **9**
6.1 The complex logarithm 9
6.2 Fractional powers and square roots 9

7 Riemann Surfaces **10**

8 A Keyhole Contour **10**

9 Summary and Problem-Solving Checklist **11**

1. From Laurent Series to Residues

Lecture 10 showed that a function holomorphic in an annulus around z_0 has a unique Laurent expansion in that annulus,

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n. \tag{1}$$

The negative powers form the principal part and classify the isolated singularity. One coefficient has a special role in integration.

Residue at an isolated singularity

The *residue* of f at z_0 is the coefficient of $(z - z_0)^{-1}$ in the Laurent expansion:

$$\boxed{\operatorname{Res}(f, z_0) = a_{-1}.} \tag{2}$$

The definition applies to removable singularities, poles, and essential singularities. At a removable singularity the residue is zero, but an essential singularity may have a finite, zero, or nonzero residue.

1.1. Why only the coefficient a_{-1} contributes

Consider the positively oriented circle

$$z - z_0 = \rho e^{i\theta}, \qquad 0 \le \theta \le 2\pi, \qquad \mathrm{d}z = i\rho e^{i\theta}\,\mathrm{d}\theta.$$

For an integer n,

$$\oint_{|z-z_0|=\rho} (z - z_0)^n \,\mathrm{d}z = i\rho^{n+1} \int_0^{2\pi} e^{i(n+1)\theta}\,\mathrm{d}\theta.$$

If $n \neq -1$, the exponential completes an integer number of oscillations and the integral vanishes. If $n = -1$, the integrand is simply i and the result is $2\pi i$. Therefore

$$\oint_{|z-z_0|=\rho} (z - z_0)^n \,\mathrm{d}z = \begin{cases} 2\pi i, & n = -1, \\ 0, & n \neq -1. \end{cases} \tag{3}$$

Integrating the Laurent series term by term gives

$$\boxed{\oint_{|z-z_0|=\rho} f(z)\,\mathrm{d}z = 2\pi i\, a_{-1} = 2\pi i \operatorname{Res}(f, z_0).} \tag{4}$$

All other Laurent coefficients cancel around the closed path. This is the local mechanism behind the residue theorem.

> The residue does not measure the order of a pole. For example, $f(z) = z^{-3}$ has a pole of order three but residue zero, whereas e^z/z^3 has the same pole order and residue $1/2$. The order comes from the most negative nonzero power; the residue comes only from the power -1.

2. Practical Calculation of Residues

Reading a_{-1} from a Laurent series is always valid, but poles permit faster formulas.

2.1. Simple poles

If z_0 is a simple pole, then

$$f(z) = \frac{a_{-1}}{z - z_0} + a_0 + a_1(z - z_0) + \cdots .$$

Multiplication by $z - z_0$ and passage to the limit gives

$$\boxed{\operatorname{Res}(f, z_0) = \lim_{z \to z_0} (z - z_0) f(z).} \tag{5}$$

Suppose $f = g/h$, where g and h are holomorphic near z_0, $h(z_0) = 0$, and $h'(z_0) \neq 0$. Taylor expansion gives

$$h(z) = h'(z_0)(z - z_0) + O((z - z_0)^2),$$

so z_0 is a simple pole provided $g(z_0) \neq 0$, and

$$\boxed{\operatorname{Res}\left(\frac{g}{h}, z_0\right) = \frac{g(z_0)}{h'(z_0)}.} \tag{6}$$

2.2. Poles of higher order

If z_0 is a pole of order m, write

$$f(z) = \frac{\phi(z)}{(z - z_0)^m},$$

where ϕ is holomorphic and $\phi(z_0) \neq 0$. The Taylor coefficient of order $m - 1$ in ϕ becomes the coefficient of $(z - z_0)^{-1}$ in f. Thus

$$\boxed{\operatorname{Res}(f, z_0) = \frac{1}{(m-1)!} \lim_{z \to z_0} \frac{\mathrm{d}^{m-1}}{\mathrm{d}z^{m-1}} \left[(z - z_0)^m f(z)\right].} \tag{7}$$

Example 11.1: simple and double poles

Let

$$f(z) = \frac{e^z}{z(z-2)^2}.$$

At $z = 0$, the pole is simple:

$$\operatorname{Res}(f, 0) = \lim_{z\to 0} \frac{e^z}{(z-2)^2} = \frac{1}{4}.$$

At $z = 2$, the pole has order two. Formula (7) gives

$$\operatorname{Res}(f, 2) = \frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{e^z}{z}\right)\bigg|_{z=2} = \frac{e^z(z-1)}{z^2}\bigg|_{z=2} = \frac{e^2}{4}.$$

The two poles of the same function require different formulas because their orders differ.

Example 11.2: extracting a residue from a series

Consider

$$F(z) = \frac{e^z}{z^3}.$$

Using $e^z = 1 + z + z^2/2 + z^3/6 + \cdots$,

$$F(z) = \frac{1}{z^3} + \frac{1}{z^2} + \frac{1}{2z} + \frac{1}{6} + \cdots.$$

Therefore

$$\boxed{\operatorname{Res}(F, 0) = \frac{1}{2}.}$$

This calculation is shorter than differentiating twice and also displays the full principal part.

Common pitfall

Before applying a pole formula, simplify common factors and determine the actual order of the singularity. A zero in the numerator may lower the pole order or make the singularity removable.

3. The Residue Theorem

The local identity (4) extends to contours enclosing several isolated singularities.

Residue theorem

Let γ be a positively oriented simple closed contour. Suppose f is holomorphic on γ and throughout its interior except at finitely many isolated singularities $z_1, \ldots, z_N$.

Then

$$\boxed{\oint_\gamma f(z)\,\mathrm{d}z = 2\pi i \sum_{k=1}^{N} \mathrm{Res}(f, z_k).} \tag{8}$$

Only singularities inside the contour contribute. Reversing the orientation changes the sign of the integral.

3.1. Proof idea

Surround each singularity z_k with a small disjoint circle C_k. Remove the interiors of these circles from the region inside γ. The remaining region contains no singularity, so the generalized Cauchy–Goursat theorem gives a zero integral around its complete boundary. The outer boundary is γ with positive orientation. Each inner boundary is traversed clockwise, so

$$0 = \oint_\gamma f(z)\,\mathrm{d}z - \sum_{k=1}^{N} \oint_{C_k} f(z)\,\mathrm{d}z.$$

Using (4) on every C_k proves (8). In this sense, a global contour integral is reconstructed from local data at the singularities.

Example 11.3: a contour containing two poles

Evaluate

$$I = \oint_{|z|=3} \frac{z}{(z-1)(z+2)}\,\mathrm{d}z.$$

Both simple poles, $z = 1$ and $z = -2$, lie inside $|z| = 3$. Their residues are

$$\mathrm{Res}(f, 1) = \frac{1}{1+2} = \frac{1}{3}, \qquad \mathrm{Res}(f, -2) = \frac{-2}{-2-1} = \frac{2}{3}.$$

Hence

$$I = 2\pi i \left(\frac{1}{3} + \frac{2}{3}\right) = \boxed{2\pi i}.$$

As a check, the partial fraction coefficients add to one, which is also the coefficient of $1/z$ in the expansion of f at infinity.

4. Real Integrals from Semicircular Contours

For an integral over the real axis, close a finite segment $[-R, R]$ with a semicircle and let $R \to \infty$. Three checks are essential:

(i) identify the poles enclosed by the chosen half-plane;

(ii) determine the orientation of the closed contour;

(iii) prove that the large-arc contribution vanishes.

4.1. A rational integral

Example 11.4: $\int_{-\infty}^{\infty}(x^2+a^2)^{-1}\,\mathrm{d}x$

Let $a>0$ and

$$f(z)=\frac{1}{z^2+a^2}=\frac{1}{(z-ia)(z+ia)}.$$

Close the contour in the upper half-plane. Only $z=ia$ lies inside, and

$$\operatorname{Res}(f,ia)=\frac{1}{2ia}.$$

On the upper semicircle Γ_R, for $R>a$,

$$|f(z)|\le\frac{1}{R^2-a^2},\qquad L(\Gamma_R)=\pi R.$$

The ML estimate gives

$$\left|\int_{\Gamma_R}f(z)\,\mathrm{d}z\right|\le\frac{\pi R}{R^2-a^2}\longrightarrow 0.$$

Therefore

$$\boxed{\int_{-\infty}^{\infty}\frac{\mathrm{d}x}{x^2+a^2}=2\pi i\left(\frac{1}{2ia}\right)=\frac{\pi}{a}.}$$

The estimate on the arc is part of the proof, not an optional remark.

4.2. The exponential determines the half-plane

For $z=x+iy$ and real k,

$$|e^{ikz}|=|e^{ikx-ky}|=e^{-ky}.$$

Thus e^{ikz} decays on an upper semicircle when $k>0$ and on a lower semicircle when $k<0$. A lower-half-plane contour closed from R to $-R$ is clockwise, so its residue-theorem contribution carries a minus sign.

Example 11.5: Fourier transform of a Lorentzian

Consider

$$I(k)=\int_{-\infty}^{\infty}\frac{e^{ikx}}{x^2+a^2}\,\mathrm{d}x,\qquad a>0.$$

For $k>0$, close above. The residue at $z=ia$ is

$$\operatorname{Res}\left(\frac{e^{ikz}}{z^2+a^2},ia\right)=\frac{e^{-ak}}{2ia},$$

so $I(k) = \pi e^{-ak}/a$. For $k < 0$, close below. The enclosed pole is $z = -ia$, and the clockwise orientation yields $I(k) = \pi e^{ak}/a$. Combining the two cases,

$$\boxed{\int_{-\infty}^{\infty} \frac{e^{ikx}}{x^2 + a^2}\,\mathrm{d}x = \frac{\pi}{a} e^{-a|k|}.} \tag{9}$$

At $k = 0$, this reduces to Example 11.4.

5. Poles on the Contour and Principal Values

The residue theorem in the form (8) requires f to be holomorphic on the contour. If a pole lies on the real integration path, the ordinary improper integral generally diverges and one must state an additional prescription.

Cauchy principal value

For a singularity at $x = x_0$ inside (a, b),

$$\boxed{\mathrm{PV}\int_a^b f(x)\,\mathrm{d}x = \lim_{\varepsilon \to 0^+} \left[\int_a^{x_0 - \varepsilon} f(x)\,\mathrm{d}x + \int_{x_0 + \varepsilon}^b f(x)\,\mathrm{d}x \right].} \tag{10}$$

The same distance is removed on both sides of the singularity. The principal value may exist even when the two one-sided improper integrals diverge.

A principal value is not the ordinary improper integral. For example, $\mathrm{PV}\int_{-1}^{1} \mathrm{d}x/x = 0$ by symmetric cancellation, while the usual improper integral does not exist because each one-sided integral diverges.

5.1. The contribution from a small indentation

Suppose f has a simple pole at x_0 with residue r_0. Near the pole,

$$f(z) = \frac{r_0}{z - x_0} + \text{a bounded term.}$$

If a real-axis contour runs from left to right and avoids the pole through a small semicircle above it, then

$$z = x_0 + \varepsilon e^{i\theta}, \qquad \theta : \pi \longrightarrow 0.$$

This indentation is clockwise. Hence

$$\int_{C_\varepsilon^+} f(z)\,\mathrm{d}z \longrightarrow \int_{\pi}^{0} \frac{r_0}{\varepsilon e^{i\theta}} i\varepsilon e^{i\theta}\,\mathrm{d}\theta = -i\pi r_0.$$

An indentation below the pole, traversed from left to right, contributes $+i\pi r_0$. The sign follows from the direction of traversal, not simply from the word “upper” or “lower.”

Example 11.6: a principal-value integral

Evaluate

$$\mathrm{PV}\int_{-\infty}^{\infty}\frac{\mathrm{d}x}{x(x^2+1)}.$$

Use an upper-half-plane contour with a small indentation above $x=0$. The ordinary pole at $z=i$ lies inside the indented contour, with

$$\mathrm{Res}(f,i)=\frac{1}{i(i+i)}=-\frac{1}{2}.$$

The residue at the indented pole is $\mathrm{Res}(f,0)=1$, so the small clockwise arc contributes $-i\pi$. The large arc vanishes. Therefore

$$\mathrm{PV}\int_{-\infty}^{\infty}f(x)\,\mathrm{d}x-i\pi=2\pi i\left(-\frac{1}{2}\right)=-i\pi,$$

and consequently

$$\boxed{\mathrm{PV}\int_{-\infty}^{\infty}\frac{\mathrm{d}x}{x(x^2+1)}=0.}$$

This agrees with the oddness of the integrand, but the contour calculation also tracks the indentation contribution explicitly.

5.2. The Sokhotski–Plemelj prescription

In the sense of distributions,

$$\boxed{\frac{1}{x-x_0\pm i0}=\mathrm{PV}\,\frac{1}{x-x_0}\mp i\pi\delta(x-x_0).} \tag{11}$$

Therefore, for a sufficiently regular test function g,

$$\int_{-\infty}^{\infty}\frac{g(x)}{x-x_0\pm i0}\,\mathrm{d}x=\mathrm{PV}\int_{-\infty}^{\infty}\frac{g(x)}{x-x_0}\,\mathrm{d}x\mp i\pi g(x_0).$$

The notation $\pm i0$ encodes a limiting displacement of the pole and fixes the imaginary contribution. It is not an algebraic decoration.

5.3. Causality and resonant particles

Linear kinetic plasma theory leads to velocity integrals of the schematic form

$$\int_{-\infty}^{\infty}\frac{g(v)}{v-\omega/k}\,\mathrm{d}v.$$

Particles with $v=\omega/k$ are resonant with the wave. A causal initial-value problem determines how the integration contour is analytically continued past this pole. The corresponding imaginary term can describe damping or growth. This is the mathematical core of the Landau prescription.

Common pitfall

The sign of a damping or growth term depends on the Fourier convention, time dependence, contour prescription, and derivative of the distribution function. Never infer the physical sign from the residue alone without stating these conventions.

6. Branch Points and Branch Cuts

Poles and isolated essential singularities are isolated points. Multivalued functions introduce a different obstruction: analytic continuation around a closed loop may return a different value.

6.1. The complex logarithm

Write $z = re^{i\theta}$ with $r > 0$. Since $e^{i(\theta+2\pi n)} = e^{i\theta}$,

$$\log z = \ln r + i(\theta + 2\pi n), \qquad n \in \mathbb{Z}. \tag{12}$$

Thus the logarithm is multivalued on $\mathbb{C} \setminus \{0\}$. A single-valued branch requires a continuous choice of argument on a restricted domain.

Principal logarithm

Choose the principal argument $-\pi < \operatorname{Arg} z < \pi$. The principal logarithm is

$$\boxed{\operatorname{Log} z = \ln |z| + i \operatorname{Arg} z,} \tag{13}$$

which is holomorphic on

$$\mathbb{C} \setminus (-\infty, 0].$$

The removed nonpositive real axis is the principal branch cut. The origin is a branch point.

For $r > 0$, the limiting values on the two sides of the negative real axis are

$$\operatorname{Log}(-r + i0) = \ln r + i\pi, \qquad \operatorname{Log}(-r - i0) = \ln r - i\pi.$$

The jump across the cut is $2\pi i$. The cut does not remove the branch point; it prevents a permitted path from encircling the branch point while remaining inside the branch domain.

6.2. Fractional powers and square roots

Define a complex power using a chosen logarithm:

$$z^{\alpha} = e^{\alpha \operatorname{Log} z}.$$

Changing the argument by 2π changes the value by the factor $e^{2\pi i\alpha}$. For $\alpha = 1/2$,

$$\sqrt{z} = \sqrt{r}\, e^{i\theta/2}.$$

After one circuit around the origin,

$$\sqrt{z} \longmapsto -\sqrt{z}.$$

After a second circuit it returns to its original value. The square root therefore has two sheets. The logarithm, which changes by $2\pi i$ after every turn, requires infinitely many sheets.

> A branch point is not a pole. There is no punctured disk around 0 on which the multivalued logarithm becomes a single-valued holomorphic function without removing an additional curve. Consequently, the isolated-singularity classification from Lecture 10 does not apply directly to the branch point.

7. Riemann Surfaces

A Riemann surface enlarges the domain so that a multivalued expression on the complex plane becomes a single-valued holomorphic function.

For $\sqrt{z}$, take two copies of a cut plane. Glue the upper bank of the cut on the first copy to the lower bank on the second, and glue the remaining two banks in the opposite pairing. Crossing the cut does not terminate the path; it moves the point to the other sheet. Two turns around the branch point return to the starting sheet.

For $\log z$, repeat this construction with infinitely many sheets. Each counterclockwise turn moves to the next sheet and adds $2\pi i$ to the value. On the resulting surface, the logarithm is single-valued even though its projection onto the punctured complex plane is not.

This geometric language matters whenever analytic continuation carries a solution through different branches. Examples include dispersion relations, wave numbers involving square roots, and special functions defined by complex integrals.

8. A Keyhole Contour

Branch cuts can be used constructively. Consider

$$I(\alpha) = \int_0^\infty \frac{x^{\alpha-1}}{1+x}\,\mathrm{d}x, \qquad 0 < \alpha < 1. \tag{14}$$

Choose

$$z^{\alpha-1} = e^{(\alpha-1)\,\mathrm{Log}\, z}, \qquad 0 < \arg z < 2\pi,$$

so the branch cut lies on the positive real axis. Integrate

$$f(z) = \frac{z^{\alpha-1}}{1+z}$$

around a keyhole contour surrounding the cut.

On the upper bank, $\arg z \to 0$ and the contribution tends to $I(\alpha)$. On the lower bank, $\arg z \to 2\pi$ and

$$z^{\alpha-1} = x^{\alpha-1}e^{2\pi i(\alpha-1)} = x^{\alpha-1}e^{2\pi i\alpha}.$$

Because the lower bank is traversed from large to small x, its contribution is $-e^{2\pi i\alpha}I(\alpha)$. The large and small circular contributions vanish for $0 < \alpha < 1$.

The only enclosed pole is $z = -1$, where $\arg(-1) = \pi$. Hence

$$\operatorname{Res}(f, -1) = (-1)^{\alpha-1} = e^{i\pi(\alpha-1)}.$$

The residue theorem gives

$$\left(1 - e^{2\pi i\alpha}\right) I(\alpha) = 2\pi i\, e^{i\pi(\alpha-1)}.$$

Using $1 - e^{2\pi i\alpha} = -2ie^{i\pi\alpha}\sin(\pi\alpha)$, we obtain

$$\boxed{\int_0^\infty \frac{x^{\alpha-1}}{1+x}\,\mathrm{d}x = \frac{\pi}{\sin(\pi\alpha)}, \qquad 0 < \alpha < 1.} \tag{15}$$

The two banks of the branch cut do not cancel because the chosen power has different phases on them. That phase difference is exactly what allows the real integral to be isolated.

9. Summary and Problem-Solving Checklist

- The residue is the Laurent coefficient a_{-1} because every other integer power integrates to zero around a closed circle.
- Simple poles, higher-order poles, and series expansions provide complementary methods for calculating residues.
- The residue theorem replaces a contour integral by a sum of local contributions from enclosed isolated singularities.
- A pole on the path requires a stated prescription. A principal value and an ordinary improper integral are different concepts.
- Branch cuts define single-valued branches on restricted domains. Riemann surfaces retain all branches on a larger domain.

Before evaluating a contour integral, check:

1. Which singularities and branch points does the integrand have?

2. Which of them lie inside or on the proposed contour?

3. Is the orientation positive or negative?

4. Does every auxiliary arc vanish in the relevant limit?

5. Have the branch and the contour prescription been stated explicitly?