GES-207-4 — Mathematical Methods in Physics

Lecture 10

Taylor and Laurent Series; Classification of Singularities

Thursday, October 22, 2026

Dr. José Paulo Marchezi — INPE

Learning goals

By the end of this lecture you should be able to:

1. derive Taylor's series from Cauchy's integral formula and explain why holomorphic functions are analytic;
2. determine the radius of convergence of a power series;
3. construct Laurent expansions in specified annuli using geometric series and partial fractions;
4. distinguish the regular and principal parts of a Laurent series;
5. classify isolated singularities as removable, poles of a given order, or essential;
6. identify the coefficient of $(z - z_0)^{-1}$ as preparation for the residue theorem.

Contents

1 From Cauchy's integral formula to Taylor series **3**

1.1 Derivation of the Taylor expansion 3

2 Power series and the radius of convergence **4**

3 Laurent series: expansions on annuli **5**

3.1 Why negative powers appear 6

4 Isolated singularities and their classification **7**

4.1 Equivalent practical tests 7

4.2 Why essential singularities are qualitatively different 8

5 Laurent expansions of rational functions in different annuli **9**

5.1 A short decision procedure 9

6 Physical interpretation: poles, natural modes, and resonances **9**

7 Summary and reference table **10**

1 From Cauchy's integral formula to Taylor series

In Lecture 09, we established that if f is holomorphic on a region containing a positively oriented simple closed contour γ and its interior, then

$$f(z) = \frac{1}{2\pi i} \oint_\gamma \frac{f(\zeta)}{\zeta - z}\, \mathrm{d}\zeta, \tag{1}$$

for every point z inside γ. The integration variable ζ moves along the contour, whereas z is the fixed point at which the function is evaluated.

The striking consequence is that complex differentiability implies a local power-series representation. Unlike a general smooth real function, a holomorphic function necessarily agrees with its Taylor series near every point of its domain.

1.1 Derivation of the Taylor expansion

Fix z_0 and choose a positively oriented circle $\gamma : |\zeta - z_0| = \rho$ whose closed interior lies in the domain of holomorphy. If $|z - z_0| < \rho$, then

$$\begin{aligned} \frac{1}{\zeta - z} &= \frac{1}{(\zeta - z_0) - (z - z_0)} \\ &= \frac{1}{\zeta - z_0} \frac{1}{1 - \dfrac{z - z_0}{\zeta - z_0}}. \end{aligned}$$

Recall the geometric series

$$\frac{1}{1 - w} = \sum_{n=0}^{\infty} w^n, \qquad |w| < 1. \tag{2}$$

On the circle, $|\zeta - z_0| = \rho$, so $|(z - z_0)/(\zeta - z_0)| < 1$. Therefore,

$$\frac{1}{\zeta - z} = \sum_{n=0}^{\infty} \frac{(z - z_0)^n}{(\zeta - z_0)^{n+1}}. \tag{3}$$

For each fixed z strictly inside the circle, this series converges uniformly with respect to ζ on γ. We may therefore integrate term by term in (1):

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \oint_\gamma f(\zeta) \sum_{n=0}^{\infty} \frac{(z - z_0)^n}{(\zeta - z_0)^{n+1}}\, \mathrm{d}\zeta \\ &= \sum_{n=0}^{\infty} \left[\frac{1}{2\pi i} \oint_\gamma \frac{f(\zeta)}{(\zeta - z_0)^{n+1}}\, \mathrm{d}\zeta \right] (z - z_0)^n. \end{aligned}$$

Cauchy's formula for derivatives states that

$$\frac{1}{2\pi i} \oint_\gamma \frac{f(\zeta)}{(\zeta - z_0)^{n+1}}\, \mathrm{d}\zeta = \frac{f^{(n)}(z_0)}{n!}.$$

Thus we have proved:

Taylor's theorem for holomorphic functions

If f is holomorphic in a neighborhood of the closed disk $\overline{D(z_0,\rho)}$, then for $|z-z_0|<\rho$,

$$\boxed{f(z)=\sum_{n=0}^{\infty}\frac{f^{(n)}(z_0)}{n!}(z-z_0)^n.} \tag{4}$$

The series converges absolutely and locally uniformly in the disk.

The same argument works for every smaller closed disk contained in the domain. Consequently, f is analytic at each point where it is holomorphic.

Important note

The largest disk contained in a chosen domain gives a *guaranteed* region of Taylor convergence, not necessarily the exact radius. Analytic continuation may allow the series to converge beyond the original domain. For a rational function, the Taylor radius is the distance from its center to the nearest genuine pole after cancellations.

2 Power series and the radius of convergence

Power series and Cauchy–Hadamard formula

A power series centered at z_0 has the form

$$\sum_{n=0}^{\infty}a_n(z-z_0)^n.$$

There exists a radius $R\in[0,\infty]$ such that the series converges absolutely for $|z-z_0|<R$ and diverges for $|z-z_0|>R$. The behavior on $|z-z_0|=R$ must be checked separately. Its radius satisfies

$$\boxed{\frac{1}{R}=\limsup_{n\to\infty}|a_n|^{1/n},}$$

with the usual conventions for $R=0$ and $R=\infty$.

When the limit exists, the ratio test may be easier:

$$\left|\frac{a_{n+1}(z-z_0)^{n+1}}{a_n(z-z_0)^n}\right|=|z-z_0|\left|\frac{a_{n+1}}{a_n}\right|.$$

If $|a_n/a_{n+1}|\to R$, the radius is R. This shortcut requires the relevant limit to exist.

Example 10.1: geometric series and the nearest singularity

Start with

$$\frac{1}{1-z}=1+z+z^2+\cdots=\sum_{n=0}^{\infty}z^n,\qquad |z|<1.$$

Here $a_n=1$, so Cauchy–Hadamard gives $1/R=1$ and $R=1$. The function has a pole at $z=1$, exactly one unit from the center $z_0=0$.

For a shifted example, consider $g(z) = 1/(3-z)$. Factor out 3:

$$g(z) = \frac{1}{3}\frac{1}{1-z/3} = \frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{z}{3}\right)^n = \sum_{n=0}^{\infty}\frac{z^n}{3^{n+1}}, \qquad |z| < 3.$$

The pole is at $z = 3$, so the radius about the origin is 3.

Example 10.2: the exponential has infinite radius

For the exponential,

$$e^z = \sum_{n=0}^{\infty}\frac{z^n}{n!}.$$

The ratio of successive terms has modulus $|z|/(n+1) \to 0$ for every fixed $z \in \mathbb{C}$. Thus $R = \infty$: the exponential is *entire*. Inserting $z = i\theta$ and separating real and imaginary parts yields

$$\begin{aligned} e^{i\theta} &= \sum_{k=0}^{\infty}\frac{(-1)^k\theta^{2k}}{(2k)!} + i\sum_{k=0}^{\infty}\frac{(-1)^k\theta^{2k+1}}{(2k+1)!} \\ &= \cos\theta + i\sin\theta. \end{aligned}$$

This recovers Euler's formula directly from convergent power series.

Common pitfalls

A Taylor series centered at z_0 is a series in powers of $(z - z_0)$, not necessarily powers of z. Also, do not infer the exact convergence radius merely from an arbitrary restricted domain: look for genuine obstructions to analytic continuation.

3 Laurent series: expansions on annuli

Taylor series contain only nonnegative powers. But a function can be holomorphic on a *punctured* region while failing to be holomorphic at its center. Negative powers provide the appropriate language.

Laurent's theorem

Suppose f is holomorphic in the annulus

$$A = \{z \in \mathbb{C} : r < |z - z_0| < R\}, \qquad 0 \le r < R \le \infty.$$

Then f has a unique Laurent expansion in that annulus:

$$\boxed{f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n.} \tag{5}$$

Its coefficients are

$$a_n = \frac{1}{2\pi i}\oint_{\gamma}\frac{f(\zeta)}{(\zeta - z_0)^{n+1}}\,\mathrm{d}\zeta, \tag{6}$$

where γ is any positively oriented circle lying in the annulus and surrounding z_0 once.

Split the series into

$$f(z) = \underbrace{\sum_{n=0}^{\infty} a_n (z - z_0)^n}_{\text{regular part}} + \underbrace{\sum_{n=1}^{\infty} \frac{a_{-n}}{(z - z_0)^n}}_{\text{principal part}}. \tag{7}$$

The regular part behaves like a Taylor series. The principal part records the singular behavior. Laurent coefficients are unique *within a specified annulus*; the same function can have different expansions in different annuli.

3.1 Why negative powers appear

The simplest example is $1/z$, which is holomorphic for $z \neq 0$ but cannot have a Taylor series centered at 0. Its Laurent expansion is simply z^{-1}, valid in $0 < |z| < \infty$.

To construct more complicated expansions, rewrite denominators so that the geometric-series variable has modulus less than one. The choice of expansion changes when we move across a singularity.

Example 10.3: two expansions of the same function

Consider $f(z) = 1/(z - 2)$, centered at the origin.
Inside the circle $|z| = 2$: factor out -2,

$$\begin{aligned} \frac{1}{z-2} &= -\frac{1}{2}\frac{1}{1 - z/2} = -\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n \\ &= -\frac{1}{2} - \frac{z}{4} - \frac{z^2}{8} - \cdots, \qquad |z| < 2. \end{aligned}$$

This is a Taylor series; it has no negative powers.
Outside the circle $|z| = 2$: factor out z,

$$\begin{aligned} \frac{1}{z-2} &= \frac{1}{z}\frac{1}{1 - 2/z} = \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{2}{z}\right)^n \\ &= \frac{1}{z} + \frac{2}{z^2} + \frac{4}{z^3} + \cdots, \qquad |z| > 2. \end{aligned}$$

This Laurent series contains negative powers. Both series represent the same function, but on different domains.

Example 10.4: partial fractions and a principal part

Take

$$h(z) = \frac{1}{z(z-3)}.$$

Seek constants A, B such that

$$\frac{1}{z(z-3)} = \frac{A}{z} + \frac{B}{z-3}.$$

Multiplying by $z(z-3)$ gives $1 = A(z-3) + Bz$. Set $z = 0$ to obtain $A = -1/3$; set $z = 3$ to obtain $B = 1/3$. Therefore,

$$h(z) = -\frac{1}{3z} + \frac{1}{3(z-3)}.$$

For $0 < |z| < 3$,

$$\frac{1}{3(z-3)} = -\frac{1}{9}\frac{1}{1-z/3} = -\frac{1}{9}\sum_{n=0}^{\infty}\left(\frac{z}{3}\right)^n.$$

Hence

$$\boxed{h(z) = -\frac{1}{3z} - \frac{1}{9} - \frac{z}{27} - \frac{z^2}{81} - \cdots, \qquad 0 < |z| < 3.} \tag{8}$$

The principal part is $-1/(3z)$, while all other terms are regular at the origin. The nearest other singularity, $z = 3$, sets the outer radius of this annulus.

Important note

A reliable Laurent-series workflow is: (1) locate singularities; (2) identify the requested annulus; (3) use partial fractions if needed; (4) factor each denominator to obtain $1/(1-w)$ with $|w| < 1$ in that annulus; (5) expand and collect positive and negative powers.

4 Isolated singularities and their classification

Isolated singularity

A point z_0 is an *isolated singularity* of f if f is holomorphic in some punctured disk $0 < |z - z_0| < R$, but is not holomorphic at z_0 as currently defined. Classification concerns whether the function admits a holomorphic extension and, if not, how it behaves near the point.

Three types of isolated singularities

Write the Laurent series in a punctured disk about z_0.

1. **Removable singularity:** every negative-power coefficient vanishes. The function extends holomorphically to z_0.
2. **Pole of order m:** the principal part is finite and its most negative nonzero power is $(z - z_0)^{-m}$.
3. **Essential singularity:** the principal part contains infinitely many nonzero negative-power terms.

4.1 Equivalent practical tests

A removable singularity is characterized by a finite limit $\lim_{z\to z_0} f(z)$; equivalently, f is bounded in a sufficiently small punctured neighborhood of z_0 (Riemann's removable singularity theorem).

A pole of order m has the form

$$f(z) = \frac{g(z)}{(z-z_0)^m}, \qquad g \text{ holomorphic near } z_0, \quad g(z_0) \neq 0. \tag{9}$$

Equivalently, $(z - z_0)^m f(z)$ extends holomorphically to z_0 with a nonzero value. At a pole, $|f(z)| \to \infty$ as $z \to z_0$.

An isolated singularity that is neither removable nor a pole is essential.

Example 10.5: a removable singularity

Consider

$$f(z) = \frac{\sin z}{z}, \qquad z \neq 0.$$

Use the Taylor series $\sin z = z - z^3/3! + z^5/5! - \cdots$:

$$\frac{\sin z}{z} = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots .$$

There are no negative powers, and $\lim_{z\to 0} \sin z/z = 1$. Define $f(0) = 1$ to obtain a holomorphic extension.

Example 10.6: determining the order of a pole

Consider

$$f(z) = \frac{e^z}{z^3}.$$

Since $e^z = 1 + z + z^2/2 + z^3/6 + \cdots$,

$$\frac{e^z}{z^3} = \frac{1}{z^3} + \frac{1}{z^2} + \frac{1}{2z} + \frac{1}{6} + \frac{z}{24} + \cdots .$$

The most negative power is z^{-3} with nonzero coefficient, so $z = 0$ is a **pole of order 3**. Equivalently, $z^3 f(z) = e^z \to 1 \neq 0$.

Example 10.7: an essential singularity

For $z \neq 0$,

$$e^{1/z} = \sum_{n=0}^{\infty} \frac{1}{n! z^n} = 1 + \frac{1}{z} + \frac{1}{2! z^2} + \frac{1}{3! z^3} + \cdots .$$

Infinitely many negative powers appear. Therefore $z = 0$ is an **essential singularity**. Along positive real $z \to 0$, $e^{1/z} \to \infty$, whereas along negative real $z \to 0$, $e^{1/z} \to 0$: there is no uniform limiting behavior.

4.2 Why essential singularities are qualitatively different

The Casorati–Weierstrass theorem states that the image of every punctured neighborhood of an essential singularity is dense in $\mathbb{C}$. Thus the function approaches values arbitrarily close to any prescribed complex number near that singularity. The stronger Great Picard theorem asserts that, with at most one exception, every complex value is assumed infinitely often in each punctured neighborhood. We will not prove these theorems here.

Common pitfalls

The expression $f(z) \to \infty$ characterizes a pole, not an essential singularity. A function can become arbitrarily large near an essential singularity without tending to infinity along every approach. Also, zeros in a numerator may cancel apparent poles: simplify or compare zero orders before classifying.

5 Laurent expansions of rational functions in different annuli

Example 10.8: a function with two singularities

Let

$$F(z) = \frac{1}{z(z-4)} = -\frac{1}{4z} + \frac{1}{4(z-4)}.$$

The singularities are $z = 0$ and $z = 4$, both simple poles. We now expand about $z_0 = 0$.
Region I: $0 < |z| < 4$**.** Because $|z/4| < 1$,

$$\frac{1}{4(z-4)} = -\frac{1}{16}\frac{1}{1-z/4} = -\frac{1}{16}\sum_{n=0}^{\infty}\left(\frac{z}{4}\right)^n.$$

Thus

$$\boxed{F(z) = -\frac{1}{4z} - \frac{1}{16} - \frac{z}{64} - \frac{z^2}{256} - \cdots, \quad 0 < |z| < 4.}$$

The principal part has one term, consistent with a simple pole at 0.
Region II: $|z| > 4$**.** Because $|4/z| < 1$,

$$\frac{1}{4(z-4)} = \frac{1}{4z}\frac{1}{1-4/z} = \frac{1}{4z}\sum_{n=0}^{\infty}\left(\frac{4}{z}\right)^n.$$

The $1/(4z)$ term cancels the $-1/(4z)$ from the partial fraction decomposition, leaving

$$\boxed{F(z) = \frac{1}{z^2} + \frac{4}{z^3} + \frac{16}{z^4} + \cdots, \quad |z| > 4.}$$

Both expansions are correct. Their different coefficients reflect different annuli, not an inconsistency.

5.1 A short decision procedure

For a rational function $N(z)/D(z)$, first factor N and D. Cancel common factors when possible, remembering that canceled points may still be removable singularities of the originally defined function. Determine the order of each remaining zero of D and the requested center. Finally, choose the appropriate geometric-series form for each factor based on the annulus.

6 Physical interpretation: poles, natural modes, and resonances

Complex-frequency response functions appear in circuits, damped oscillators, wave propagation, and plasma physics. Consider a forced damped oscillator:

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = F_0 e^{-i\omega t}.$$

Seeking a steady response $x(t) = X(\omega)e^{-i\omega t}$ gives

$$(-\omega^2 - 2i\gamma\omega + \omega_0^2)X(\omega) = F_0.$$

Hence the transfer function is

$$H(\omega) = \frac{X(\omega)}{F_0} = \frac{1}{\omega_0^2 - \omega^2 - 2i\gamma\omega}. \tag{10}$$

Its poles are the roots of the denominator:

$$\omega_{\pm} = -i\gamma \pm \sqrt{\omega_0^2 - \gamma^2}.$$

For $0 < \gamma < \omega_0$, these poles lie below the real frequency axis. They encode damped natural modes; when a real driving frequency approaches the real part of a weakly damped pole, the response can become large. Near a simple pole ω_p,

$$H(\omega) \approx \frac{a_{-1}}{\omega - \omega_p} + \text{regular terms}.$$

The coefficient a_{-1} is the *residue*. It controls the leading singular contribution and will be central to contour integration in the next lecture.

Important note

In plasma physics, response functions and analytically continued dispersion relations also involve complex frequencies. Poles can encode normal modes or resonant response, but their precise physical meaning depends on the model, the quantity being continued, and the chosen causality convention. Not every zero or pole of a dielectric response has the same interpretation.

7 Summary and reference table

Concept	**Mathematical signature**	**Interpretation**
Taylor series	$\sum_{n\geq 0} a_n(z-z_0)^n$	Holomorphic behavior near z_0
Laurent series	$\sum_{n\in\mathbb{Z}} a_n(z-z_0)^n$	Holomorphic on an annulus
Removable singularity	No negative powers	Holomorphic extension exists
Pole of order m	Lowest nonzero power $-m$	Divergence like $(z-z_0)^{-m}$
Essential singularity	Infinitely many negative powers	Highly nonuniform behavior
Residue	Coefficient a_{-1}	Key quantity for next lecture

The conceptual progression of Lectures 09 and 10 is

Cauchy's formula $\Longrightarrow$ Taylor series $\Longrightarrow$ Laurent series $\Longrightarrow$ singularities and residues.