GES-207-4 — Mathematical Methods in Physics

Lecture 09

Elementary Complex Functions, Contour Integration, and Cauchy's Theorem

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Learning goals

This lecture combines a focused review of elementary complex functions with the central results of complex integration. By the end, you should be able to:

1. use the complex exponential and a chosen branch of the logarithm;
2. explain why the topology of a domain matters for contour integrals;
3. parameterize a contour and evaluate complex line integrals;
4. state Cauchy–Goursat's theorem and understand its connection with the Cauchy–Riemann equations;
5. recognize when an integral is path independent;
6. derive and apply Cauchy's integral formula and its derivative formula;
7. obtain Cauchy's estimates and prove Liouville's theorem.

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1 From complex differentiability to complex integration

In Lecture 08, a complex function $f = u + iv$ was called holomorphic when its derivative exists in every complex direction. The Cauchy–Riemann equations,

$$u_x = v_y, \qquad u_y = -v_x,$$

encode the compatibility between those directions. Here we discover a much stronger consequence: integrals of holomorphic functions around suitable closed contours vanish. Even more surprisingly, values and derivatives inside a contour can be reconstructed from values on the contour itself.

The central progression is

$$\boxed{\text{holomorphy} \longrightarrow \text{Cauchy's theorem} \longrightarrow \text{Cauchy's integral formula} \longrightarrow \text{Liouville.}}$$

2 The complex exponential and logarithm

2.1 Exponential: periodicity and geometry

Complex exponential

For $z = x + iy$, define

$$\boxed{e^z = e^x(\cos y + i \sin y).}$$

It is entire (holomorphic on all of $\mathbb{C}$), $(e^z)' = e^z$, and

$$e^{z+w} = e^z e^w, \qquad e^{z+2\pi i} = e^z.$$

The modulus and argument are easy to read:

$$|e^{x+iy}| = e^x, \qquad \arg(e^{x+iy}) \equiv y \pmod{2\pi}.$$

Thus vertical lines $x = c$ map onto circles of radius e^c, whereas horizontal lines $y = c$ map onto rays of argument c. Because of periodicity, e^z cannot be one-to-one on the entire plane.

Example 9.1: different inputs, identical output

Compute

$$e^{1+i\pi} = e(\cos \pi + i \sin \pi) = -e, \qquad e^{1+3\pi i} = -e.$$

The two inputs differ by $2\pi i$. In general, $e^z = e^w$ precisely when $z - w = 2\pi i k$ for some $k \in \mathbb{Z}$.

2.2 Why the logarithm has branches

To solve $e^w = z \neq 0$, write $z = re^{i\theta}$. If $w = a + ib$, then $e^a = r$ and $b = \theta + 2\pi k$. Consequently,

$$\boxed{\log z = \{\ln |z| + i(\theta + 2\pi k) : k \in \mathbb{Z}\}.}$$

This is a set of values rather than a globally single-valued function.

Principal logarithm and branch cut

Choose $\operatorname{Arg} z \in (-\pi, \pi)$ on the slit plane $D = \mathbb{C} \setminus (-\infty, 0]$. The **principal logarithm** on D is

$$\boxed{\operatorname{Log} z = \ln|z| + i \operatorname{Arg} z, \qquad z \in D.}$$

It is holomorphic there and $(\operatorname{Log} z)' = 1/z$.

The negative real axis is removed because the argument jumps by 2π when one crosses it. No single-valued holomorphic logarithm exists on all of $\mathbb{C} \setminus \{0\}$: going once around the origin changes a continuously tracked argument by 2π.

Example 9.2: a principal square root

Let $z = 1 + i$. Then $|z| = \sqrt{2}$ and $\operatorname{Arg} z = \pi/4$. Hence

$$\operatorname{Log}(1 + i) = \tfrac{1}{2} \ln 2 + i \tfrac{\pi}{4}.$$

Defining the principal square root by $z^{1/2} = e^{\operatorname{Log} z/2}$ gives

$$(1 + i)^{1/2} = 2^{1/4} \left(\cos \frac{\pi}{8} + i \sin \frac{\pi}{8} \right).$$

Squaring doubles the angle and squares the modulus, recovering $1 + i$. The other square root is its negative.

What we need from this review

The logarithm is locally well behaved, but its global definition depends on the domain. This is the first sign that *holes in a domain* can affect integration, even when a function is holomorphic at every point of that domain.

3 Curves, orientation, and simply connected domains

A **contour** is a continuous, piecewise continuously differentiable map $\gamma : [a, b] \to \mathbb{C}$. It is **closed** when $\gamma(a) = \gamma(b)$. A simple closed contour does not intersect itself except at the common endpoints.

A domain $\Omega \subseteq \mathbb{C}$ is open and connected. Informally, it is **simply connected** if every closed loop in Ω can be continuously contracted to a point without leaving Ω. A disk is simply connected; a punctured disk and $\mathbb{C} \setminus \{0\}$ are not.

Common pitfalls

Holomorphic *on a contour* is not enough to invoke the simplest form of Cauchy's theorem. One must also control the region enclosed by the contour. A singularity may lie inside even when the integrand is perfectly regular along the boundary.

4 Complex contour integrals

4.1 Definition and calculation recipe

Contour integral

Let f be continuous along a piecewise smooth contour $\gamma : [a, b] \to \Omega$. Then

$$\boxed{\int_\gamma f(z)\,\mathrm{d}z = \int_a^b f(\gamma(t))\gamma'(t)\,\mathrm{d}t.}$$

To calculate a contour integral, choose a parameterization $z = \gamma(t)$, calculate $\mathrm{d}z = \gamma'(t)\,\mathrm{d}t$, substitute both into the integrand, and integrate with respect to the real parameter t.

Reversing orientation changes the sign:

$$\int_{-\gamma} f(z)\,\mathrm{d}z = -\int_\gamma f(z)\,\mathrm{d}z.$$

Integrals are additive when a contour is assembled from consecutive pieces.

Example 9.3: integration along a line segment

Evaluate $\int_\gamma z^2\,\mathrm{d}z$ from 0 to $1 + i$ along the straight segment. Parameterize

$$\gamma(t) = (1 + i)t, \qquad \gamma'(t) = 1 + i, \qquad 0 \le t \le 1.$$

Then

$$\begin{aligned}\int_\gamma z^2\,\mathrm{d}z &= \int_0^1 (1 + i)^2 t^2 (1 + i)\,\mathrm{d}t \\ &= (1 + i)^3 \int_0^1 t^2\,\mathrm{d}t = \frac{(1 + i)^3}{3} = \frac{-2 + 2i}{3}.\end{aligned}$$

This also equals $F(1 + i) - F(0)$ for the antiderivative $F(z) = z^3/3$.

4.2 A closed circle and a nonzero integral

For a counterclockwise circle $|z| = R$, use

$$\gamma(t) = Re^{it}, \qquad \gamma'(t) = iRe^{it}, \qquad 0 \le t \le 2\pi.$$

Example 9.4: integrating $1/z$

Direct substitution yields

$$\oint_{|z|=R} \frac{1}{z}\,\mathrm{d}z = \int_0^{2\pi} \frac{iRe^{it}}{Re^{it}}\,\mathrm{d}t = i\int_0^{2\pi} \mathrm{d}t = \boxed{2\pi i}.$$

The answer is independent of $R > 0$, but changes sign if the circle is traversed clockwise.

More generally, for every integer n,

$$\oint_{|z|=R} z^n\,\mathrm{d}z = iR^{n+1}\int_0^{2\pi} e^{i(n+1)t}\,\mathrm{d}t = \begin{cases} 2\pi i, & n = -1, \\ 0, & n \ne -1. \end{cases}$$

This calculation anticipates the residue theorem: the coefficient of z^{-1} is special.

4.3 Connection with real line integrals

Write $f(z) = u(x,y) + iv(x,y)$ and $\mathrm{d}z = \mathrm{d}x + i\,\mathrm{d}y$. Multiplying,

$$\begin{aligned} f(z)\,\mathrm{d}z &= (u+iv)(\,\mathrm{d}x + i\,\mathrm{d}y) \\ &= (u\,\mathrm{d}x - v\,\mathrm{d}y) + i(v\,\mathrm{d}x + u\,\mathrm{d}y). \end{aligned}$$

Therefore,

$$\boxed{\oint_\gamma f(z)\,\mathrm{d}z = \oint_\gamma (u\,\mathrm{d}x - v\,\mathrm{d}y) + i\oint_\gamma (v\,\mathrm{d}x + u\,\mathrm{d}y).}$$

This decomposition connects complex integration to Green's theorem in the plane.

5 Cauchy–Goursat's theorem

Cauchy–Goursat theorem

If $\Omega \subseteq \mathbb{C}$ is simply connected and f is holomorphic throughout Ω, then for every piecewise smooth closed contour $\gamma \subset \Omega$,

$$\boxed{\oint_\gamma f(z)\,\mathrm{d}z = 0.}$$

For a simple closed contour, it is enough that f be holomorphic on an open neighborhood of the contour and its enclosed region.

5.1 Derivation using Green's theorem

We first prove the result under stronger regularity assumptions: assume u, v have continuous first partial derivatives on a neighborhood of a bounded region D with positively oriented, piecewise smooth, simple boundary $\gamma = \partial D$.

Green's theorem states

$$\oint_{\partial D} P\,\mathrm{d}x + Q\,\mathrm{d}y = \iint_D (Q_x - P_y)\,\mathrm{d}A.$$

Apply it separately to the real and imaginary parts of the complex integral.

Real part. Choose $P = u$ and $Q = -v$:

$$\oint_\gamma (u\,\mathrm{d}x - v\,\mathrm{d}y) = \iint_D (-v_x - u_y)\,\mathrm{d}A.$$

By the Cauchy–Riemann equation $u_y = -v_x$, the integrand vanishes.

Imaginary part. Choose $P = v$ and $Q = u$:

$$\oint_\gamma (v\,\mathrm{d}x + u\,\mathrm{d}y) = \iint_D (u_x - v_y)\,\mathrm{d}A.$$

By the other Cauchy–Riemann equation $u_x = v_y$, this integrand also vanishes. Hence

$$\boxed{\oint_\gamma f(z)\,\mathrm{d}z = 0.}$$

Why the name Cauchy–Goursat?

The Green's theorem argument above is an instructive proof under additional differentiability assumptions. Goursat's theorem establishes the conclusion assuming only complex differentiability (holomorphy), without separately assuming continuous derivatives. We use the full theorem as a stated result; its general proof requires a different argument.

5.2 Why $1/z$ does not contradict Cauchy's theorem

The function $1/z$ is holomorphic on $\mathbb{C} \setminus \{0\}$, but not at $z = 0$. The circle $|z| = R$ encloses the missing point, and the punctured plane is not simply connected. Thus the hypotheses of the theorem are not satisfied. The nonzero result $2\pi i$ is fully consistent with Cauchy–Goursat.

6 Path independence and complex antiderivatives

Antiderivative and path independence

An antiderivative of f on Ω is a holomorphic function F such that $F' = f$. If F exists, then for every piecewise smooth contour from z_a to z_b,

$$\boxed{\int_\gamma f(z)\,\mathrm{d}z = F(z_b) - F(z_a).}$$

In a simply connected domain, every holomorphic function has an antiderivative, and its contour integrals depend only on the endpoints.

To see why, suppose γ_1 and γ_2 have the same endpoints. Following γ_1 and then the reverse of γ_2 forms a closed contour, so Cauchy's theorem gives

$$0 = \int_{\gamma_1} f(z)\,\mathrm{d}z - \int_{\gamma_2} f(z)\,\mathrm{d}z.$$

The integrals are therefore equal.

Example 9.5: a path-independent integral

Let $f(z) = 3z^2 + 2z$ and take any contour from 0 to i. Since f is entire, $F(z) = z^3 + z^2$ is an antiderivative. Thus

$$\int_\gamma (3z^2 + 2z)\,\mathrm{d}z = F(i) - F(0) = i^3 + i^2 = -i - 1.$$

No parameterization is needed once an antiderivative is known.

Common pitfalls

Do not confuse the existence of a *local* antiderivative with the existence of a *global* one. The function $1/z$ has local logarithmic antiderivatives, but no single-valued holomorphic antiderivative on the entire punctured plane.

7 Cauchy's integral formula

Cauchy's integral formula

Let γ be a positively oriented, piecewise smooth, simple closed contour. Suppose f is holomorphic on an open neighborhood of γ and its interior, and let z_0 lie inside γ. Then

$$\boxed{f(z_0) = \frac{1}{2\pi i} \oint_\gamma \frac{f(z)}{z - z_0}\, \mathrm{d}z.}$$

The striking message is that values on a boundary determine every value in the interior.

7.1 Where the factor $2\pi i$ comes from

Begin with the identity

$$\frac{f(z)}{z - z_0} = \frac{f(z) - f(z_0)}{z - z_0} + \frac{f(z_0)}{z - z_0}.$$

Let C_ε be a small positively oriented circle centered at z_0, contained inside γ. The integrand $f(z)/(z - z_0)$ is holomorphic in the region between γ and C_ε. Applying Cauchy's theorem to that region (with the inner boundary oriented clockwise) gives

$$\oint_\gamma \frac{f(z)}{z - z_0}\, \mathrm{d}z = \oint_{C_\varepsilon} \frac{f(z)}{z - z_0}\, \mathrm{d}z.$$

On the small circle, split the integral:

$$\begin{aligned} \oint_{C_\varepsilon} \frac{f(z)}{z - z_0}\, \mathrm{d}z &= \oint_{C_\varepsilon} \frac{f(z) - f(z_0)}{z - z_0}\, \mathrm{d}z + f(z_0) \oint_{C_\varepsilon} \frac{1}{z - z_0}\, \mathrm{d}z \\ &= E_\varepsilon + 2\pi i f(z_0). \end{aligned}$$

Since f is continuous at z_0,

$$|E_\varepsilon| \le 2\pi \max_{|z - z_0| = \varepsilon} |f(z) - f(z_0)| \longrightarrow 0.$$

Taking $\varepsilon \to 0$ yields

$$\oint_\gamma \frac{f(z)}{z - z_0}\, \mathrm{d}z = 2\pi i f(z_0),$$

which is exactly the formula.

The logical difference between two theorems

Cauchy's theorem concerns integrals of a function holomorphic *throughout* the enclosed region. Cauchy's integral formula deliberately introduces a factor $1/(z - z_0)$ with a singularity inside. The formula evaluates that nonzero integral using the value $f(z_0)$.

7.2 Worked contour-integral examples

Example 9.6: a simple pole inside a circle

Evaluate

$$I = \oint_{|z|=3} \frac{\cos z}{z - i}\, \mathrm{d}z$$

with counterclockwise orientation. The numerator $f(z) = \cos z$ is entire, and $z_0 = i$ lies inside the circle since $|i| = 1 < 3$. Cauchy's integral formula gives

$$I = 2\pi i\, f(i) = 2\pi i \cos(i) = \boxed{2\pi i \cosh 1}.$$

If the contour were clockwise, the answer would be its negative.

Example 9.7: a pole outside the contour

Consider

$$J = \oint_{|z|=1} \frac{1}{z - 2}\, \mathrm{d}z.$$

The singularity $z = 2$ is outside the closed unit disk. Therefore $1/(z - 2)$ is holomorphic on a neighborhood of the disk, and Cauchy's theorem implies

$$\boxed{J = 0.}$$

The location of the singularity, not just the algebraic appearance of the denominator, determines which theorem applies.

8 Cauchy's formula for derivatives

Higher-derivative integral formula

Under the same hypotheses as Cauchy's integral formula, every derivative of f exists and

$$\boxed{f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_{\gamma} \frac{f(z)}{(z - z_0)^{n+1}}\, \mathrm{d}z, \qquad n = 0, 1, 2, \ldots.}$$

The first derivative formula can be motivated by differentiating the kernel $1/(z - z_0)$ with respect to z_0; each differentiation raises the denominator power and produces a factorial factor. Justification requires uniform control on compact subsets inside the contour.

Example 9.8: a third-order denominator

Evaluate

$$K = \oint_{|z|=2} \frac{e^z}{z^3}\, \mathrm{d}z.$$

Here $z_0 = 0$, $f(z) = e^z$, and $(z - z_0)^3$ corresponds to $n + 1 = 3$, hence $n = 2$. Since $f''(0) = 1$,

$$K = \frac{2\pi i}{2!} f''(0) = \boxed{\pi i}.$$

9 The ML bound and Cauchy's estimates

For a piecewise smooth contour of length L and a continuous function satisfying $|f(z)| \le M$ along it, the triangle inequality yields

$$\boxed{\left|\int_\gamma f(z)\,\mathrm{d}z\right| \le ML.}$$

This follows from the parameterized integral:

$$\left|\int_a^b f(\gamma(t))\gamma'(t)\,\mathrm{d}t\right| \le \int_a^b |f(\gamma(t))|\,|\gamma'(t)|\,\mathrm{d}t \le ML.$$

Cauchy's estimates

If f is holomorphic on a neighborhood of the closed disk $|z - z_0| \le R$, and $|f(z)| \le M_R$ on the circle $|z - z_0| = R$, then

$$\boxed{|f^{(n)}(z_0)| \le \frac{n!M_R}{R^n}.}$$

Indeed, on the circle $|z - z_0| = R$, the modulus of the denominator in the derivative formula is R^{n+1}, and the contour length is $2\pi R$. Thus

$$|f^{(n)}(z_0)| \le \frac{n!}{2\pi}\frac{M_R}{R^{n+1}}(2\pi R) = \frac{n!M_R}{R^n}.$$

Example 9.9: estimating derivatives from boundary values

Suppose f is holomorphic on a neighborhood of $|z| \le 2$ and $|f(z)| \le 5$ on $|z| = 2$. Cauchy's estimate for $n = 2$ gives

$$|f''(0)| \le \frac{2! \cdot 5}{2^2} = \boxed{\frac{5}{2}}.$$

We obtained a bound on an interior derivative without knowing an explicit formula for f.

10 Liouville's theorem and the fundamental theorem of algebra

Liouville's theorem

Every bounded entire function is constant.

Proof. Suppose f is entire and $|f(z)| \le M$ for all $z \in \mathbb{C}$. Fix any $z_0 \in \mathbb{C}$. Because f is entire, Cauchy's estimate holds on circles of *arbitrarily large* radius R:

$$|f'(z_0)| \le \frac{M}{R}.$$

Letting $R \to \infty$ gives $f'(z_0) = 0$. Since z_0 was arbitrary, $f' \equiv 0$ on the connected plane, so f is constant. □

Example 9.10: fundamental theorem of algebra

Every nonconstant complex polynomial has at least one complex zero. To see why, suppose a polynomial $p(z) = a_n z^n + \cdots + a_0$, $n \ge 1$, has no zeros. Then $g(z) = 1/p(z)$ is entire. Since $|p(z)| \to \infty$ as $|z| \to \infty$, we have $g(z) \to 0$ at infinity. On a sufficiently large closed disk, g is

continuous and therefore bounded; outside that disk it is also bounded. Thus g is a bounded entire function, so Liouville's theorem makes g constant. This contradicts the assumption that p is nonconstant. Hence p has a zero.

10.1 A useful extension: polynomial growth

Cauchy's estimates also explain why growth bounds constrain entire functions. If $|f(z)| \leq C(1 + |z|^m)$ for all z, where m is a nonnegative integer, then on $|z| = R$,

$$|f^{(n)}(0)| \leq \frac{n!C(1 + R^m)}{R^n}.$$

For every integer $n > m$, the right-hand side tends to zero as $R \to \infty$, so $f^{(n)}(0) = 0$. Since entire functions have a globally convergent Taylor expansion, only terms through degree m remain: f is a polynomial of degree at most m.

11 Physical perspective and conceptual summary

In two-dimensional electrostatics and idealized potential flows, harmonic functions describe source-free scalar potentials. The real and imaginary parts of a holomorphic function are harmonic; their level curves are locally orthogonal wherever the derivative is nonzero. Complex contour methods provide exact information about analytic fields and boundary values. The integral formula expresses a particularly strong form of interior control by boundary data.

Concept	**Meaning**
Complex logarithm	Locally analytic inverse of the exponential; globally branch-dependent.
Contour integral	Integral calculated using $z = \gamma(t)$ and $\mathrm{d}z = \gamma'(t)\,\mathrm{d}t$.
Simply connected domain	No holes obstructing contraction of closed loops.
Cauchy–Goursat	Closed integrals of holomorphic functions vanish under the stated domain hypotheses.
Antiderivative	Makes contour integrals depend only on endpoints.
Cauchy integral formula	Determines interior values from boundary values.
Cauchy estimates	Bound interior derivatives using boundary magnitudes.
Liouville	Bounded entire functions must be constant.

Common pitfalls

- Check whether a singularity lies *inside* the contour, not merely on it.
- Check orientation: clockwise traversal reverses the sign.
- A branch of Log is not holomorphic across its branch cut.
- Cauchy's theorem and Cauchy's integral formula have different integrands and purposes.
- The Green's theorem proof uses additional regularity; the full Cauchy–Goursat theorem does not.

- In the derivative formula, $(z - z_0)^{n+1}$ corresponds to the nth derivative.

Core formulas

$$
\begin{aligned}
e^{x+iy} &= e^x(\cos y + i \sin y),\\
\operatorname{Log} z &= \ln|z| + i \operatorname{Arg} z \quad \text{(on the chosen branch)},\\
\int_\gamma f(z)\,\mathrm{d}z &= \int_a^b f(\gamma(t))\gamma'(t)\,\mathrm{d}t,\\
\oint_\gamma f(z)\,\mathrm{d}z &= 0 \quad \text{(Cauchy–Goursat hypotheses)},\\
f(z_0) &= \frac{1}{2\pi i}\oint_\gamma \frac{f(z)}{z - z_0}\,\mathrm{d}z,\\
f^{(n)}(z_0) &= \frac{n!}{2\pi i}\oint_\gamma \frac{f(z)}{(z - z_0)^{n+1}}\,\mathrm{d}z,\\
|f^{(n)}(z_0)| &\le \frac{n! M_R}{R^n}.
\end{aligned}
$$