GES-207-4 — Mathematical Methods in Physics

Lecture 08

Complex Numbers and Analytic Functions: The Cauchy–Riemann Conditions

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Learning goals

This lecture begins the module on complex analysis. Its main objective is to explain why differentiability with respect to a complex variable is much more restrictive than differentiability with respect to two real variables. By the end, you should be able to:

1. perform arithmetic with complex numbers and convert between Cartesian and polar forms;
2. use Euler's and De Moivre's formulas to compute powers and roots;
3. interpret limits and derivatives in the complex plane;
4. derive the Cauchy–Riemann equations and state precisely when they imply complex differentiability;
5. distinguish complex differentiability at a point from holomorphy on an open set;
6. recognize the geometric meaning of a nonzero complex derivative;
7. verify harmonicity and construct harmonic conjugates;
8. relate analytic functions to two-dimensional electrostatic potentials.

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1 Why complex analysis is different

In previous lectures, complex numbers appeared as scalars in vector spaces and in Hermitian inner products. We now study functions for which both the input and output are complex:

$$f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}.$$

A complex number contains two real coordinates, so any such function may also be viewed as a map from a subset of $\mathbb{R}^2$ into $\mathbb{R}^2$. However, the requirement that the function possess a *complex derivative* imposes additional relationships between its two real components. These relationships are the Cauchy–Riemann equations. Their consequences are surprisingly strong: complex-differentiable functions on open sets are smooth, admit local power-series representations, and have harmonic real and imaginary parts.

2 The algebra of complex numbers

Complex numbers and their operations

The field of complex numbers is

$$\mathbb{C} = \{x + iy : x, y \in \mathbb{R},\ i^2 = -1\}.$$

For $z = x + iy$, define $\operatorname{Re} z = x$, $\operatorname{Im} z = y$, its conjugate $\bar{z} = x - iy$, and its modulus $|z| = \sqrt{x^2 + y^2}$. Addition and multiplication are given by

$$\begin{aligned}(x_1 + iy_1) + (x_2 + iy_2) &= (x_1 + x_2) + i(y_1 + y_2),\\ (x_1 + iy_1)(x_2 + iy_2) &= (x_1x_2 - y_1y_2) + i(x_1y_2 + y_1x_2).\end{aligned}$$

The conjugate satisfies $z\bar{z} = x^2 + y^2 = |z|^2$, so for $z \neq 0$,

$$\boxed{\frac{1}{z} = \frac{\bar{z}}{|z|^2}.}$$

This identity explains the standard division rule: multiply numerator and denominator by the conjugate of the denominator. Other useful identities are $\overline{zw} = \bar{z}\bar{w}$, $|zw| = |z||w|$, and $|z + w| \leq |z| + |w|$.

Example 8.1: Complex division and modulus

Compute $(2 + 3i)/(1 - i)$. Multiplying by $1 + i$ gives

$$\frac{2 + 3i}{1 - i} = \frac{(2 + 3i)(1 + i)}{(1 - i)(1 + i)} = \frac{-1 + 5i}{2} = -\frac{1}{2} + \frac{5}{2}i.$$

Its modulus is $\sqrt{(1/2)^2 + (5/2)^2} = \sqrt{26}/2$. The same result follows from $|2 + 3i|/|1 - i| = \sqrt{13}/\sqrt{2}$.

Geometrically, $\mathbb{C}$ is identified with the Euclidean plane $\mathbb{R}^2$. The modulus is the distance from the origin, and $|z - w|$ is the distance between two points. Complex multiplication will turn out to combine rotations and scalings.

3 Polar representation, Euler's formula, and roots

If $z = x + iy \neq 0$, define $r = |z|$ and choose an angle θ such that $x = r\cos\theta$ and $y = r\sin\theta$. The angle is determined only up to integer multiples of 2π:

$$\arg z = \theta + 2\pi k, \qquad k \in \mathbb{Z}.$$

The *principal argument* $\operatorname{Arg} z \in (-\pi, \pi]$ selects one representative. The origin has no argument.

Euler's formula,

$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta,}$$

leads to the polar form $z = re^{i\theta}$. Multiplication now becomes especially transparent:

$$(r_1 e^{i\theta_1})(r_2 e^{i\theta_2}) = r_1 r_2 e^{i(\theta_1+\theta_2)}.$$

Thus, moduli multiply and arguments add. A rigorous derivation of Euler's formula from power series will appear later in this module.

De Moivre's formula and nth roots

For $n \in \mathbb{Z}$ and $z = re^{i\theta} \neq 0$,

$$\boxed{z^n = r^n(\cos(n\theta) + i\sin(n\theta)).}$$

If $w = \rho e^{i\phi} \neq 0$ and n is a positive integer, the solutions of $z^n = w$ are

$$\boxed{z_k = \rho^{1/n} e^{i(\phi+2\pi k)/n}, \qquad k = 0, \ldots, n-1.}$$

To derive the root formula, write $z = re^{i\theta}$. From $z^n = w$ we obtain $r^n = \rho$ and $n\theta = \phi + 2\pi k$. The distinct roots lie on a circle of radius $\rho^{1/n}$, separated by angles $2\pi/n$.

Example 8.2: Cube roots of unity

Solving $z^3 = 1$ gives

$$z_0 = 1, \qquad z_1 = -\frac{1}{2} + i\frac{\sqrt{3}}{2}, \qquad z_2 = -\frac{1}{2} - i\frac{\sqrt{3}}{2}.$$

These three points form an equilateral triangle on the unit circle. Their rotational symmetry is representative of the symmetries encountered in oscillation modes and wave problems.

A subtlety about arguments

The expressions $\arg z$ and $\operatorname{Arg} z$ are not interchangeable: $\arg z$ denotes all possible arguments, whereas $\operatorname{Arg} z$ is a particular branch. This distinction becomes essential when studying the complex logarithm and branch cuts.

4 Limits and continuity in the complex plane

Complex limit

For $f : \Omega \to \mathbb{C}$ and an accumulation point z_0 of Ω, we write $\lim_{z\to z_0} f(z) = L$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$0 < |z - z_0| < \delta,\ z \in \Omega \quad \Longrightarrow \quad |f(z) - L| < \varepsilon.$$

Continuity at $z_0 \in \Omega$ means $\lim_{z\to z_0} f(z) = f(z_0)$.

Unlike the real line, the complex plane has infinitely many directions of approach. A limit exists only when all approaches give the same value. Showing that two paths yield different limits is enough to prove that a limit does not exist.

Example 8.3: A path-dependent limit

Consider $g(z) = \bar{z}/z$ for $z \neq 0$. Along the real axis, $z = t \neq 0$ gives $g(t) = 1$. Along the imaginary axis, $z = it$ gives $g(it) = (-it)/(it) = -1$. Consequently,

$$\lim_{z\to 0} \frac{\bar{z}}{z} \quad \text{does not exist.}$$

The two paths approach the same point but produce incompatible values.

5 Complex differentiability and holomorphy

Complex derivative and holomorphic function

Let $\Omega \subseteq \mathbb{C}$ be open and $f : \Omega \to \mathbb{C}$. The function is *complex differentiable at* $z_0 \in \Omega$ if

$$\boxed{f'(z_0) = \lim_{h\to 0} \frac{f(z_0 + h) - f(z_0)}{h}}$$

exists and is independent of how the nonzero complex increment h tends to zero. The function is *holomorphic on* Ω if it is complex differentiable at every point of Ω. A function holomorphic on all of $\mathbb{C}$ is called *entire*.

Differentiability at a single point does not automatically imply holomorphy in a neighborhood. The latter is a property of an open set. As in real calculus, differentiability at a point implies continuity there, because

$$f(z_0 + h) - f(z_0) = h\frac{f(z_0 + h) - f(z_0)}{h} \longrightarrow 0.$$

For a real function, one takes a difference quotient along a one-dimensional axis. For a complex function, the same quotient must converge for every complex increment, which is a much stronger demand.

6 Deriving the Cauchy–Riemann equations

Write

$$f(x + iy) = u(x, y) + iv(x, y),$$

where u and v are real-valued functions of two real variables. Let $z_0 = x_0 + iy_0$. If $f'(z_0)$ exists, we may compute it along two particular families of increments.

6.1 Approach along the real direction

Take $h = \Delta x \in \mathbb{R}$. Then

$$\begin{aligned} f'(z_0) &= \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x + iy_0) - f(x_0 + iy_0)}{\Delta x} \\ &= \lim_{\Delta x \to 0} \left[\frac{u(x_0 + \Delta x, y_0) - u(x_0, y_0)}{\Delta x} + i \frac{v(x_0 + \Delta x, y_0) - v(x_0, y_0)}{\Delta x} \right] \\ &= u_x(x_0, y_0) + iv_x(x_0, y_0). \end{aligned}$$

6.2 Approach along the imaginary direction

Now take $h = i\Delta y$, where $\Delta y \in \mathbb{R}$. Since $1/i = -i$,

$$\begin{aligned} f'(z_0) &= \lim_{\Delta y \to 0} \frac{f(x_0 + i(y_0 + \Delta y)) - f(x_0 + iy_0)}{i\Delta y} \\ &= \frac{1}{i}(u_y(x_0, y_0) + iv_y(x_0, y_0)) \\ &= v_y(x_0, y_0) - iu_y(x_0, y_0). \end{aligned}$$

Because both paths must produce the same complex derivative, their real and imaginary parts must agree.

Cauchy–Riemann theorem: necessary and sufficient conditions

If $f = u + iv$ is complex differentiable at $z_0 = x_0 + iy_0$, then the partial derivatives exist there and satisfy

$$\boxed{u_x = v_y, \qquad u_y = -v_x.}$$

In this case,

$$\boxed{f'(z_0) = u_x + iv_x = v_y - iu_y.}$$

Conversely, if u and v are *real differentiable at* (x_0, y_0) and satisfy these equations there, then f is complex differentiable at z_0. In particular, it suffices that u, v be C^1 in a neighborhood of z_0 and satisfy the equations at z_0.

6.3 Why the converse works

Real differentiability means that for $h = a + ib$,

$$\begin{aligned} u(x_0 + a, y_0 + b) - u(x_0, y_0) &= u_x a + u_y b + o(|h|), \\ v(x_0 + a, y_0 + b) - v(x_0, y_0) &= v_x a + v_y b + o(|h|). \end{aligned}$$

Using $u_y = -v_x$ and $v_y = u_x$, the increment of f becomes

$$\begin{aligned} f(z_0 + h) - f(z_0) &= (u_x + iv_x)a + (-v_x + iu_x)b + o(|h|) \\ &= (u_x + iv_x)(a + ib) + o(|h|) \\ &= (u_x + iv_x)h + o(|h|). \end{aligned}$$

Dividing by $h \neq 0$ gives

$$\frac{f(z_0+h)-f(z_0)}{h} = u_x + iv_x + \frac{o(|h|)}{h} \longrightarrow u_x + iv_x.$$

The error term tends to zero in modulus because $|o(|h|)/h| = o(1)$.

Common pitfalls

The Cauchy–Riemann equations alone, checked at one point, are *not* sufficient without a regularity assumption such as real differentiability. Also, complex differentiability at one point is weaker than holomorphy on an open neighborhood.

7 The Jacobian and the geometry of an analytic map

Viewing f as a real map $(x, y) \mapsto (u, v)$, its Jacobian is

$$Df(x,y) = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}.$$

At a point where Cauchy–Riemann holds, this matrix has the special form

$$\boxed{Df = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}, \qquad a = u_x, \quad b = v_x.}$$

Write $a + ib = \rho e^{i\theta}$ with $\rho = \sqrt{a^2+b^2} = |f'(z_0)|$. Then

$$Df = \rho \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

Thus, when $f'(z_0) \neq 0$, the linear approximation is a rotation through θ followed by a uniform scaling by ρ. Such maps preserve angles between smooth curves locally; this is the geometric meaning of *conformality*. At points where $f' = 0$, the linear map vanishes and this conclusion does not apply.

8 Worked examples of complex differentiability

Example 8.4: The entire function $f(z) = z^2$

Expanding $(x+iy)^2$ gives

$$u = x^2 - y^2, \qquad v = 2xy.$$

The derivatives are $u_x = 2x$, $u_y = -2y$, $v_x = 2y$, and $v_y = 2x$. Hence $u_x = v_y$ and $u_y = -v_x$ everywhere. Since these functions are smooth, f is entire, with

$$\boxed{f'(z) = u_x + iv_x = 2x + 2iy = 2z.}$$

Alternatively, the difference quotient is $((z+h)^2 - z^2)/h = 2z + h \to 2z$.

Example 8.5: Complex conjugation is nowhere differentiable

For $f(z) = \bar{z} = x - iy$, we have $u = x$, $v = -y$. Then $u_x = 1$ and $v_y = -1$, so Cauchy–Riemann fails at every point. Equivalently,

$$\frac{\overline{z+h} - \bar{z}}{h} = \frac{\bar{h}}{h}$$

is 1 for real h and -1 for purely imaginary h. Therefore f is nowhere complex differentiable, although it is a smooth real map $\mathbb{R}^2 \to \mathbb{R}^2$.

Example 8.6: A function differentiable at only one point

For $f(z) = |z|^2 = x^2 + y^2$, we have $u = x^2 + y^2$, $v = 0$. The Cauchy–Riemann equations become $2x = 0$ and $2y = 0$, so they hold only at the origin. The components are smooth; therefore f is complex differentiable at $z = 0$. Directly,

$$f'(0) = \lim_{h \to 0} \frac{|h|^2}{h} = \lim_{h \to 0} \bar{h} = 0.$$

At every $z \neq 0$ the Cauchy–Riemann equations fail. Consequently, f is not holomorphic on any nonempty open set.

9 Harmonic functions and harmonic conjugates

Harmonic function

A twice continuously differentiable real-valued function u on an open subset of $\mathbb{R}^2$ is *harmonic* if it satisfies Laplace's equation

$$\boxed{\Delta u = u_{xx} + u_{yy} = 0.}$$

Real and imaginary parts of holomorphic functions

If $f = u + iv$ is holomorphic on Ω, then u and v are harmonic on Ω. The function v is called a *harmonic conjugate* of u when $u + iv$ is holomorphic.

Holomorphic functions are smooth; differentiating Cauchy–Riemann and commuting mixed derivatives yields

$$\Delta u = u_{xx} + u_{yy} = (v_y)_x + (-v_x)_y = v_{yx} - v_{xy} = 0,$$
$$\Delta v = v_{xx} + v_{yy} = (-u_y)_x + (u_x)_y = -u_{yx} + u_{xy} = 0.$$

The gradients of the harmonic conjugates are orthogonal:

$$\nabla u \cdot \nabla v = u_x v_x + u_y v_y = u_x v_x - v_x u_x = 0.$$

Where both gradients are nonzero, their level curves therefore meet at right angles.

Local versus global harmonic conjugates

Every harmonic function has a harmonic conjugate *locally*. On a simply connected domain it has a globally defined harmonic conjugate, unique up to an additive real constant. On a domain with holes, a global single-valued conjugate need not exist. This distinction will matter when studying complex integration and logarithms.

9.1 A systematic method for constructing a harmonic conjugate

Given a harmonic function $u(x, y)$, use

$$v_x = -u_y, \qquad v_y = u_x.$$

First integrate one equation with respect to its variable, including an arbitrary function of the other variable. Then differentiate the result and compare with the second equation to determine that function.

Example 8.7: Finding a harmonic conjugate

Let $u(x, y) = x^2 - y^2 + 2y$. We first verify harmonicity:

$$u_{xx} = 2, \qquad u_{yy} = -2, \qquad \Delta u = 0.$$

The Cauchy–Riemann equations require

$$v_x = -u_y = 2y - 2, \qquad v_y = u_x = 2x.$$

Integrating the first equation with respect to x gives

$$v(x, y) = (2y - 2)x + g(y).$$

Differentiating with respect to y gives $v_y = 2x + g'(y)$. Comparing with $v_y = 2x$ shows $g'(y) = 0$, hence $g(y) = C$. Therefore,

$$\boxed{v(x, y) = 2xy - 2x + C,}$$

and $f = u + iv = z^2 - 2iz + iC$ is holomorphic. The arbitrary real constant in v does not affect the derivative.

10 Physical application: two-dimensional electrostatics

In electrostatics, $\mathbf{E} = -\nabla\phi$. In a homogeneous charge-free region, Gauss's law $\nabla \cdot \mathbf{E} = 0$ implies

$$\boxed{\Delta\phi = 0.}$$

Thus the electrostatic potential is harmonic. Locally we may find a harmonic conjugate ψ and form an analytic complex potential $F(z) = \phi(x, y) + i\psi(x, y)$. The Cauchy–Riemann equations show that

$$\nabla\phi \cdot \nabla\psi = 0.$$

Therefore the level curves of ϕ (equipotentials) and ψ (field-line curves, wherever gradients do not vanish) are orthogonal. Indeed, $\mathbf{E}$ is parallel to $-\nabla\phi$ and tangent to the curves $\psi =$ constant.

Example 8.8: Potential of an infinite charged wire

For an infinite line charge of density λ, the electrostatic potential outside the wire, relative to a reference radius r_0, can be written

$$\phi(r) = -\frac{\lambda}{2\pi\varepsilon_0} \ln\left(\frac{r}{r_0}\right).$$

On any domain that avoids the origin and admits a single-valued argument, choose a branch of the logarithm $\operatorname{Log} z = \ln|z| + i \operatorname{Arg} z$. A complex potential is

$$F(z) = -\frac{\lambda}{2\pi\varepsilon_0} \operatorname{Log} z + \text{real constant}.$$

Its real part gives ϕ up to the choice of reference, and its imaginary part is

$$\psi(x, y) = -\frac{\lambda}{2\pi\varepsilon_0} \operatorname{Arg} z.$$

The equipotentials $r =$ constant are circles. The curves $\operatorname{Arg} z =$ constant are radial rays, tangent to the radial electric field

$$\mathbf{E} = -\nabla\phi = \frac{\lambda}{2\pi\varepsilon_0 r} \hat{\mathbf{r}}.$$

The logarithm cannot be defined as a single-valued holomorphic function on the entire punctured plane $\mathbb{C} \setminus \{0\}$; a branch cut or a local domain is required.

The same mathematical construction is useful in idealized two-dimensional fluid flows and magnetic potential problems. The physical interpretation depends on the governing equations and the chosen complex potential.

11 Conceptual summary

Concept	Meaning
Complex plane	$z = x + iy$ represents a point of $\mathbb{R}^2$ with complex multiplication.
Polar form	$z = re^{i\theta}$ separates magnitude from angle.
Complex limit	The same limit must hold along every approach path.
Complex derivative	The quotient $(f(z+h) - f(z))/h$ has a direction-independent limit.
Holomorphic function	Complex differentiable throughout an open set.
Cauchy–Riemann	$u_x = v_y$ and $u_y = -v_x$; necessary, and sufficient with real differentiability.
Conformal map	When $f' \neq 0$, its differential is a rotation and scaling.
Harmonic function	Solves $\Delta u = 0$.
Harmonic conjugate	v such that $u + iv$ is holomorphic.

Common pitfalls

- Do not check the complex derivative along only one direction.
- Do not assume that satisfying Cauchy–Riemann at a point alone proves differentiability without a regularity hypothesis.
- Do not confuse differentiability at one point with holomorphy on an open set.

- Do not forget the arbitrary function when integrating v_x or v_y to find a harmonic conjugate.
- Do not treat $\operatorname{Arg} z$ or $\operatorname{Log} z$ as globally single-valued analytic functions on the punctured plane.

Core formulas

$$z = x + iy = re^{i\theta}, \quad \bar{z} = x - iy, \quad |z|^2 = z\bar{z}.$$

$$z_k = \rho^{1/n} e^{i(\phi+2\pi k)/n}, \qquad k = 0, \dots, n-1.$$

$$f'(z_0) = \lim_{h\to 0} \frac{f(z_0+h) - f(z_0)}{h}.$$

$$\boxed{u_x = v_y, \quad u_y = -v_x, \quad f' = u_x + iv_x.}$$

$$\boxed{\Delta u = \Delta v = 0, \qquad \nabla u \cdot \nabla v = 0.}$$

12 Preparation problems for the next class

Two problems for board presentation

Solve the following two problems using the concepts and methods developed in this chapter. Show the intermediate steps and explain why your conclusions follow. Two students will be selected to present their reasoning at the board (approximately 10 minutes each). Solutions are intentionally not included.

Problem 1 — Cauchy–Riemann conditions

Verify whether $f(z) = e^{x}(\cos y + i \sin y)$ satisfies the Cauchy–Riemann conditions throughout $\mathbb{C}$. If it does, compute $f'(z)$ using the formula $f' = u_x + iv_x$.

Problem 2 — Harmonic conjugate

For $u(x, y) = x^2 - y^2 + x$, verify that u is harmonic ($\Delta u = 0$). Find a harmonic conjugate $v(x, y)$ such that $f = u + iv$ is holomorphic, using the Cauchy–Riemann equations $v_x = -u_y$ and $v_y = u_x$.