GES-207-4 — Mathematical Methods in Physics

Lecture 07
Vector Calculus and Tensor Algebra

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Learning goals

This lecture develops the mathematical language used to describe spatially varying physical fields and direction-dependent material responses. By the end of these notes, you should be able to:

1. distinguish scalar and vector fields and interpret the nabla operator;
2. calculate gradients, directional derivatives, divergence, curl, and the Laplacian;
3. connect local derivatives to flux and circulation through Gauss's and Stokes's theorems;
4. recognize the vector-calculus structure of Maxwell's equations;
5. apply Einstein summation notation, the Kronecker delta, and the Levi–Civita symbol;
6. interpret second-order tensors as linear maps and construct dyadic products;
7. decompose a tensor into symmetric and antisymmetric parts and compute its trace;
8. explain directional projectors and simple anisotropic conductivity and pressure tensors.

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1 Scalar and vector fields

In earlier lectures, vectors and linear transformations were studied largely independently of position. In continuum physics, however, a quantity often varies from point to point. A *field* assigns a physical or mathematical quantity to every point of a spatial region $\Omega \subseteq \mathbb{R}^3$.

Scalar and vector fields

A *scalar field* is a function $\phi : \Omega \to \mathbb{R}$, written $\phi = \phi(x, y, z)$. A *vector field* is a function $\mathbf{F} : \Omega \to \mathbb{R}^3$ with Cartesian representation

$$\boxed{\mathbf{F}(x, y, z) = F_x(x, y, z)\hat{\mathbf{e}}_x + F_y(x, y, z)\hat{\mathbf{e}}_y + F_z(x, y, z)\hat{\mathbf{e}}_z.}$$

Temperature, pressure, and electrostatic potential are familiar scalar fields. Electric and magnetic fields and fluid velocity are vector fields. A scalar field supplies one number at each point; a vector field supplies a magnitude and direction. A field may also depend on time, but our initial differential operations concern its spatial dependence.

2 The nabla operator

In Cartesian coordinates with a fixed orthonormal basis, define the differential operator

$$\boxed{\nabla = \hat{\mathbf{e}}_x \frac{\partial}{\partial x} + \hat{\mathbf{e}}_y \frac{\partial}{\partial y} + \hat{\mathbf{e}}_z \frac{\partial}{\partial z}.}$$

Coordinate convention

Throughout this chapter, spatial derivatives are taken in a fixed orthonormal Cartesian basis. The formulas for divergence and curl require additional geometric factors in curvilinear coordinates.

The symbol ∇ is pronounced *nabla* or *del*. It is an operator, not an ordinary numerical vector: its components act by differentiation on whatever follows it. Three important combinations are

$$\underbrace{\nabla\phi}_{\text{gradient: vector}}, \qquad \underbrace{\nabla \cdot \mathbf{F}}_{\text{divergence: scalar}}, \qquad \underbrace{\nabla \times \mathbf{F}}_{\text{curl: vector}}.$$

These operations are defined below and should be distinguished by both their algebraic output and their geometric interpretation.

3 Gradient and directional derivatives

Gradient

For a differentiable scalar field ϕ, its gradient is

$$\boxed{\nabla\phi = \frac{\partial\phi}{\partial x}\hat{\mathbf{e}}_x + \frac{\partial\phi}{\partial y}\hat{\mathbf{e}}_y + \frac{\partial\phi}{\partial z}\hat{\mathbf{e}}_z.}$$

The gradient describes the first-order spatial variation of a scalar field. For a small displacement $\mathrm{d}\mathbf{r}$, the total differential is $\mathrm{d}\phi = \nabla\phi \cdot \mathrm{d}\mathbf{r}$. Hence the gradient converts the displacement into the corresponding first-order change in ϕ.

Example 7.1: computing a gradient

Consider $\phi(x, y, z) = x^2 + 2y^2 + 3z^2$. Differentiating with respect to one coordinate while holding the others fixed gives

$$\frac{\partial \phi}{\partial x} = 2x, \qquad \frac{\partial \phi}{\partial y} = 4y, \qquad \frac{\partial \phi}{\partial z} = 6z.$$

Therefore

$$\boxed{\nabla \phi = (2x, 4y, 6z), \qquad \nabla \phi(1, 1, 1) = (2, 4, 6).}$$

For a unit vector $\hat{\mathbf{n}}$, the *directional derivative* is

$$\boxed{D_{\hat{\mathbf{n}}} \phi = \hat{\mathbf{n}} \cdot \nabla \phi.}$$

Writing $\hat{\mathbf{n}} \cdot \nabla \phi = \|\nabla \phi\| \cos \theta$ shows that the largest possible directional derivative is $\|\nabla \phi\|$, attained when $\hat{\mathbf{n}}$ points along the gradient (provided the gradient is nonzero). At $(1, 1, 1)$ in Example 7.1, the rates of change along the x and y axes are 2 and 4, while the maximum rate is $\sqrt{2^2 + 4^2 + 6^2} = \sqrt{56}$.

The gradient is also normal to regular level surfaces $\phi(x, y, z) = c$. To see why, take a differentiable curve $\mathbf{r}(t)$ lying on the surface. Because $\phi(\mathbf{r}(t)) = c$, the chain rule gives

$$0 = \frac{\mathrm{d}}{\mathrm{d}t} \phi(\mathbf{r}(t)) = \nabla \phi \cdot \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}.$$

The tangent $\mathrm{d}\mathbf{r}/\mathrm{d}t$ is therefore perpendicular to $\nabla \phi$.

4 Divergence and local flux

Divergence

For a differentiable vector field $\mathbf{F} = (F_x, F_y, F_z)$,

$$\boxed{\nabla \cdot \mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}.}$$

Divergence measures the net outward flux per unit volume in the limit of a shrinking region. A positive value indicates locally source-like behavior; a negative value indicates sink-like behavior. A zero divergence means no *net* infinitesimal outward flux, not necessarily a zero field.

Example 7.2: a radial field

For $\mathbf{F} = (x, y, z)$,

$$\nabla \cdot \mathbf{F} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 1 + 1 + 1 = \boxed{3}.$$

For the inward field $\mathbf{G} = (-x, -y, -z)$, the divergence is -3. Both results agree with the local outward/inward flux interpretation.

5 Curl and local circulation

Curl

The curl of a differentiable vector field is

$$\boxed{\nabla \times \mathbf{F} = \begin{vmatrix} \hat{\mathbf{e}}_x & \hat{\mathbf{e}}_y & \hat{\mathbf{e}}_z \\ \partial_x & \partial_y & \partial_z \\ F_x & F_y & F_z \end{vmatrix}.}$$

Here the determinant is a mnemonic for the Cartesian component formula:

$$\begin{aligned} \nabla \times \mathbf{F} = {} & (\partial_y F_z - \partial_z F_y)\hat{\mathbf{e}}_x \\ & + (\partial_z F_x - \partial_x F_z)\hat{\mathbf{e}}_y \\ & + (\partial_x F_y - \partial_y F_x)\hat{\mathbf{e}}_z. \end{aligned}$$

The curl quantifies the local circulation density of the field. Its direction is determined by the right-hand rule.

Example 7.3: planar rotation

Let $\mathbf{F} = (-y, x, 0)$. The three curl components are

$$\begin{aligned} (\nabla \times \mathbf{F})_x &= \partial_y 0 - \partial_z x = 0, \\ (\nabla \times \mathbf{F})_y &= \partial_z(-y) - \partial_x 0 = 0, \\ (\nabla \times \mathbf{F})_z &= \partial_x x - \partial_y(-y) = 1 - (-1) = 2. \end{aligned}$$

Thus $\boxed{\nabla \times \mathbf{F} = 2\hat{\mathbf{e}}_z}$. The vector field circulates counterclockwise in the xy plane when viewed from positive z.

6 Differential identities and the Laplacian

If the required second partial derivatives are continuous, mixed derivatives commute. Consequently,

$$\boxed{\nabla \times (\nabla \phi) = \mathbf{0}, \qquad \nabla \cdot (\nabla \times \mathbf{F}) = 0.}$$

For example, the x component of the first identity is

$$(\nabla \times \nabla \phi)_x = \partial_y \partial_z \phi - \partial_z \partial_y \phi = 0.$$

The remaining components follow identically. The *scalar Laplacian* is the divergence of the gradient:

$$\boxed{\nabla^2 \phi = \nabla \cdot \nabla \phi = \partial_x^2 \phi + \partial_y^2 \phi + \partial_z^2 \phi.}$$

For the function in Example 7.1, $\nabla^2 \phi = 2 + 4 + 6 = 12$. The Laplacian appears in Laplace's and Poisson's equations, diffusion, and wave propagation.

7 Integral theorems: from local to global

Differential operators describe local properties. Integral theorems relate them to fluxes and circulation over finite regions.

Divergence theorem (Gauss)

For a sufficiently smooth vector field on a regular volume V, with outward-oriented boundary ∂V,

$$\boxed{\iiint_V (\nabla \cdot \mathbf{F})\,\mathrm{d}V = \iint_{\partial V} \mathbf{F} \cdot \mathrm{d}\mathbf{S}.}$$

The volume integral measures total divergence inside V; the surface integral measures net outward flux across its boundary.

Example 7.4: Gauss's theorem on a unit cube

Let $V = [0,1]^3$ and $\mathbf{F} = (x, y, z)$. Since $\nabla \cdot \mathbf{F} = 3$,

$$\iiint_V 3\,\mathrm{d}V = 3.$$

On the faces $x = 1$, $y = 1$, and $z = 1$, the outward flux is 1 per face. On $x = 0$, $y = 0$, and $z = 0$, the normal component vanishes. Thus the total outward flux is $1 + 1 + 1 = 3$, in agreement with the theorem.

Stokes's theorem

For a sufficiently smooth oriented surface S with positively oriented boundary ∂S,

$$\boxed{\iint_S (\nabla \times \mathbf{F}) \cdot \mathrm{d}\mathbf{S} = \oint_{\partial S} \mathbf{F} \cdot \mathrm{d}\boldsymbol{\ell}.}$$

The left side measures the flux of the curl through S; the right side measures circulation around its boundary. The orientations must be compatible by the right-hand rule.

Example 7.5: Stokes's theorem on a disk

Take $\mathbf{F} = (-y, x, 0)$ and the unit disk in the xy plane with normal $+\hat{\mathbf{e}}_z$. Since $\nabla \times \mathbf{F} = 2\hat{\mathbf{e}}_z$, the surface integral is 2π. Parameterize the positively oriented boundary by $\mathbf{r}(t) = (\cos t, \sin t, 0)$, $0 \le t \le 2\pi$. Then

$$\mathbf{F}(\mathbf{r}(t)) = (-\sin t, \cos t, 0) = \mathbf{r}'(t),$$

so

$$\oint_{\partial S} \mathbf{F} \cdot \mathrm{d}\boldsymbol{\ell} = \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\,\mathrm{d}t = \int_0^{2\pi} 1\,\mathrm{d}t = 2\pi.$$

Both sides agree.

8 A physical application: Maxwell's equations

In SI units, the differential form of Maxwell's equations is

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}, \qquad \nabla \cdot \mathbf{B} = 0,$$
$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}.$$

The divergence equations relate local sources to fields; the curl equations relate circulation to time variation and electric current. Their role here is to illustrate the mathematical language rather than to derive electromagnetic theory.

9 Index notation and Einstein summation

Cartesian components of a vector are written v_i, where $i = 1, 2, 3$. The scalar product of two real vectors is

$$\mathbf{v} \cdot \mathbf{w} = \sum_{i=1}^{3} v_i w_i.$$

In *Einstein summation convention*, an index repeated exactly twice in a term is summed over its range:

$$\boxed{\mathbf{v} \cdot \mathbf{w} = v_i w_i.}$$

For complex vectors, the Hermitian inner product instead reads $\langle \mathbf{v}, \mathbf{w} \rangle = \overline{v_i} w_i$ under the convention used in earlier lectures.

For a matrix T, the matrix–vector product becomes

$$\boxed{w_i = T_{ij} v_j = \sum_{j=1}^{3} T_{ij} v_j.}$$

Here i is a *free index*: it labels the component of the output. The index j is a *dummy index*: it is summed over and disappears from the result. Free indices must match on both sides of an equation.

Example 7.6: expanding repeated indices

Let

$$T = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix}, \qquad \mathbf{v} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}.$$

Then

$$\begin{aligned} w_1 &= T_{11} v_1 + T_{12} v_2 + T_{13} v_3 = 2(1) + 1(2) + 0(1) = 4, \\ w_2 &= T_{21} v_1 + T_{22} v_2 + T_{23} v_3 = 1(1) + 3(2) + 0(1) = 7, \\ w_3 &= T_{31} v_1 + T_{32} v_2 + T_{33} v_3 = 0(1) + 0(2) + 4(1) = 4. \end{aligned}$$

Therefore $\boxed{\mathbf{w} = (4, 7, 4)^T}$.

10 Kronecker delta and Levi–Civita symbol

The *Kronecker delta* is

$$\boxed{\delta_{ij} = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}}$$

It represents the identity matrix in an orthonormal Cartesian basis and satisfies

$$\delta_{ij} v_j = v_i, \qquad \hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_j = \delta_{ij}.$$

For example, with $i = 2$, the expression $\delta_{2j} v_j = 0v_1 + 1v_2 + 0v_3 = v_2$.

The *Levi–Civita symbol* ε_{ijk} is $+1$ for even permutations of $(1, 2, 3)$, -1 for odd permutations, and 0 whenever two indices coincide. Thus

$$\varepsilon_{123} = +1, \qquad \varepsilon_{213} = -1, \qquad \varepsilon_{113} = 0.$$

The cross product can be expressed as

$$\boxed{(\mathbf{a}\times\mathbf{b})_i = \varepsilon_{ijk}a_j b_k.}$$

For $i=1$, the only nonzero terms are $\varepsilon_{123}a_2b_3$ and $\varepsilon_{132}a_3b_2$, yielding $(\mathbf{a}\times\mathbf{b})_1 = a_2b_3 - a_3b_2$.

11 Vector calculus in index notation

Write $\partial_i = \partial/\partial x_i$. The three fundamental operations become

$$\boxed{(\nabla\phi)_i = \partial_i\phi, \qquad \nabla\cdot\mathbf{F} = \partial_i F_i, \qquad (\nabla\times\mathbf{F})_i = \varepsilon_{ijk}\partial_j F_k.}$$

These formulas demonstrate why index notation is so useful: it expresses component operations without repeatedly writing long Cartesian expansions. In the last expression, the free index i identifies the curl component; j and k are summed.

12 Second-order tensors and linear maps

A scalar has no component index, a vector has one, and a second-order tensor has two:

$$\phi, \qquad v_i, \qquad T_{ij}.$$

In a three-dimensional orthonormal Cartesian basis, a second-order tensor is represented by a 3×3 matrix,

$$T = \begin{pmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{pmatrix}.$$

A particularly useful interpretation is that a second-order tensor acts linearly on vectors:

$$\boxed{\mathbf{w} = T\mathbf{v}, \qquad w_i = T_{ij}v_j.}$$

The matrix represents the tensor in a chosen basis; the geometric object is not tied to one coordinate representation. Under a change between orthonormal Cartesian bases described by an orthogonal matrix Q, the component matrix changes as $T' = QTQ^T$ when vector components transform as $\mathbf{v}' = Q\mathbf{v}$. This transformation rule distinguishes tensor components from an arbitrary table of numbers.

13 Dyadic products and tensor symmetry

Given two vectors $\mathbf{a}$ and $\mathbf{b}$, their *dyadic* or *outer* product is

$$\boxed{(\mathbf{a}\otimes\mathbf{b})_{ij} = a_i b_j, \qquad \mathbf{a}\otimes\mathbf{b} = \mathbf{a}\mathbf{b}^T.}$$

For instance, if $\mathbf{a} = (1,2,0)^T$ and $\mathbf{b} = (3,0,1)^T$,

$$\mathbf{a}\otimes\mathbf{b} = \begin{pmatrix} 3 & 0 & 1 \\ 6 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}.$$

A tensor is *symmetric* if $T_{ij} = T_{ji}$, equivalently $T^T = T$; it is *antisymmetric* if $T_{ij} = -T_{ji}$, equivalently $T^T = -T$. Every real second-order tensor admits the decomposition

$$\boxed{T = \underbrace{\tfrac{1}{2}(T + T^T)}_{T^{(s)}} + \underbrace{\tfrac{1}{2}(T - T^T)}_{T^{(a)}}.}$$

Indeed, $(T^{(s)})^T = T^{(s)}$ and $(T^{(a)})^T = -T^{(a)}$. The symmetric and antisymmetric parts often describe distinct physical effects.

14 Trace and contraction

The *trace* is the sum of diagonal components:

$$\boxed{\operatorname{tr} T = T_{ii} = T_{11} + T_{22} + T_{33}.}$$

This is an example of *index contraction*: summing over a repeated pair reduces the number of free indices. Thus $T_{ij}v_j$ has one free index and represents a vector, whereas T_{ii} has none and represents a scalar. In Example 7.6, $\operatorname{tr} T = 2 + 3 + 4 = 9$.

15 Anisotropic conductivity and directional projectors

In an isotropic conducting medium, Ohm's law may be written $\mathbf{J} = \sigma\mathbf{E}$, where σ is a scalar conductivity. The current is parallel to the electric field. In an anisotropic medium the response depends on direction and is represented by a tensor:

$$\boxed{\mathbf{J} = \boldsymbol{\sigma}\mathbf{E}, \qquad J_i = \sigma_{ij}E_j.}$$

A general conductivity tensor need not make $\mathbf{J}$ parallel to $\mathbf{E}$.

Suppose a unit vector $\hat{\mathbf{b}}$ defines a preferred direction, such as a magnetic-field direction. The parallel and perpendicular projectors are

$$\boxed{P_\parallel = \hat{\mathbf{b}}\hat{\mathbf{b}}^T, \qquad P_\perp = I - \hat{\mathbf{b}}\hat{\mathbf{b}}^T.}$$

For any vector $\mathbf{v}$,

$$P_\parallel \mathbf{v} = (\hat{\mathbf{b}} \cdot \mathbf{v})\hat{\mathbf{b}}, \qquad P_\perp \mathbf{v} = \mathbf{v} - (\hat{\mathbf{b}} \cdot \mathbf{v})\hat{\mathbf{b}}.$$

These are orthogonal projections: $P_\parallel^2 = P_\parallel$, $P_\perp^2 = P_\perp$, and $P_\parallel P_\perp = 0$. A simple axisymmetric tensor with different parallel and perpendicular responses can be written

$$\boxed{T = T_\parallel P_\parallel + T_\perp P_\perp.}$$

This is a direct application of the projector and spectral-decomposition ideas developed earlier in the course. It represents a particularly simple anisotropic response, not every possible tensor.

16 Pressure tensor in a magnetized plasma

In an isotropic fluid, the pressure tensor is $\mathbf{P} = pI$, or $P_{ij} = p\delta_{ij}$. In a gyrotropic magnetized plasma, pressures may differ parallel and perpendicular to the magnetic-field direction. A standard representation is

$$\boxed{\mathbf{P} = p_\parallel \hat{\mathbf{b}}\hat{\mathbf{b}}^T + p_\perp(I - \hat{\mathbf{b}}\hat{\mathbf{b}}^T).}$$

If $p_\parallel = p_\perp = p$, then $\mathbf{P} = pI$. Thus the isotropic result is recovered as a special case. The tensor form encodes directional dependence without having to choose coordinates aligned with the magnetic field.

17 A unified viewpoint: the gradient of a vector field

The bridge between vector calculus and tensor algebra becomes explicit when we differentiate a vector field. Define the velocity-gradient tensor, using the convention adopted here, by

$$\boxed{(\nabla \mathbf{v})_{ij} = \partial_j v_i.}$$

Because the expression has two free indices, it is a second-order tensor in Cartesian coordinates. Its trace is

$$\boxed{\mathrm{tr}(\nabla \mathbf{v}) = \partial_i v_i = \nabla \cdot \mathbf{v}.}$$

For example, if $\mathbf{v} = (x, 2y, 3z)$, then

$$\nabla \mathbf{v} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}, \qquad \mathrm{tr}(\nabla \mathbf{v}) = 1 + 2 + 3 = 6 = \nabla \cdot \mathbf{v}.$$

This shows that the divergence is not an unrelated operation: it is the trace of a tensor describing spatial derivatives of a vector field.

Common pitfalls

- The gradient acts on a scalar field and produces a vector; divergence acts on a vector field and produces a scalar.
- A vanishing divergence does not mean that the vector field vanishes.
- A curl determinant is a Cartesian mnemonic, not an ordinary determinant of numbers.
- The orientation of the boundary curve in Stokes's theorem must agree with the chosen surface normal.
- In Einstein notation, a summed index appears twice in a term; free indices must agree on both sides of an equation.
- A tensor is a geometric linear map; its matrix components depend on the chosen basis.

18 Summary and key formulas

The three differential operators provide complementary descriptions of spatial variation:

Operation	Output	Interpretation
$\nabla \phi$	vector	steepest increase
$\nabla \cdot \mathbf{F}$	scalar	local net flux
$\nabla \times \mathbf{F}$	vector	local circulation

Their integral counterparts are Gauss's and Stokes's theorems. Tensor notation packages vector and matrix operations into expressions such as $w_i = T_{ij} v_j$, and the Kronecker delta and Levi–Civita symbol compactly represent identity and orientation. A second-order tensor is naturally understood as a linear map whose components depend on the basis; tensor projectors describe physically preferred directions.

Essential relations

$$(\nabla\phi)_i = \partial_i\phi, \qquad \nabla\cdot\mathbf{F} = \partial_i F_i, \qquad (\nabla\times\mathbf{F})_i = \varepsilon_{ijk}\partial_j F_k,$$
$$\nabla^2\phi = \partial_i\partial_i\phi, \qquad \delta_{ij}v_j = v_i, \qquad w_i = T_{ij}v_j,$$
$$(\mathbf{a}\otimes\mathbf{b})_{ij} = a_i b_j, \qquad \operatorname{tr} T = T_{ii},$$
$$T^{(s)} = \tfrac{1}{2}(T + T^T), \qquad T^{(a)} = \tfrac{1}{2}(T - T^T),$$
$$P_\parallel = \hat{\mathbf{b}}\hat{\mathbf{b}}^T, \qquad P_\perp = I - P_\parallel.$$