GES-207-4 — Mathematical Methods in Physics

Lecture 06

Hilbert Spaces: Completeness, L^2, and Infinite Orthonormal Bases

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Dr. José Paulo Marchezi — INPE

Learning goals

This lecture closes the first module by extending inner products, norms, orthogonality, projection, and orthonormal bases to infinite-dimensional spaces. By the end of these notes, you should be able to:

1. distinguish convergence from the Cauchy property;
2. explain completeness and define a Hilbert space;
3. use tails of convergent series to prove that sequences in ℓ^2 are Cauchy and to identify their limits;
4. define $L^1([a,b])$ and $L^2([a,b])$, compare them on finite intervals, and test power singularities with improper integrals;
5. explain equality almost everywhere and use indicator functions;
6. compute L^2 inner products, norms, and distances;
7. construct finite orthogonal projections and verify their best-approximation property;
8. derive Bessel's inequality and Parseval's identity;
9. compute complex Fourier coefficients and analyze finite Fourier modes;
10. distinguish pointwise convergence from L^2 convergence.

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1 The central question

In a finite-dimensional space, an expansion in a basis is a finite sum. In an infinite-dimensional space, it becomes a sequence of partial sums. Before writing

$$f = \sum_{k=1}^{\infty} c_k u_k,$$

we must answer three separate questions:

1. In which norm should the partial sums converge?
2. Do the coefficients produce a Cauchy sequence of partial sums?
3. If they do, does the limit still belong to the original space?

The third question leads to completeness. Completeness is what prevents a well-behaved sequence from converging toward an object missing from the space.

Guiding chain of ideas

partial sums $\Longrightarrow$ Cauchy sequences $\Longrightarrow$ completeness
$\Longrightarrow$ infinite expansions whose limits remain in the space

2 From finite to infinite dimensions

Let $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ be an orthonormal basis of $\mathbb{R}^n$ or $\mathbb{C}^n$. Every vector satisfies

$$\mathbf{v} = \sum_{k=1}^{n} \langle \mathbf{e}_k, \mathbf{v} \rangle \, \mathbf{e}_k$$

and

$$\|\mathbf{v}\|^2 = \sum_{k=1}^{n} | \langle \mathbf{e}_k, \mathbf{v} \rangle |^2.$$

The first formula decomposes a vector into components; the second is the Pythagorean theorem.

In infinite dimensions, the analogous expression is a series,

$$f = \sum_{k=1}^{\infty} c_k u_k.$$

The symbol on the right means the limit of the partial sums

$$S_N = \sum_{k=1}^{N} c_k u_k.$$

Thus an infinite expansion is a statement about convergence, not a purely algebraic sum.

3 Norm convergence, Cauchy sequences, and completeness

Convergence in norm

A sequence $\{v_n\}$ in a normed space V converges to $v \in V$ if, for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$n \geq N \quad \Longrightarrow \quad \|v_n - v\| < \varepsilon.$$

We write $v_n \to v$.

For example, in $\mathbb{R}^2$,

$$v_n = \begin{pmatrix} 1 + 1/n \\ 2 - 1/n \end{pmatrix} \longrightarrow v = \begin{pmatrix} 1 \\ 2 \end{pmatrix},$$

because $\|v_n - v\| = \sqrt{2}/n \to 0$.

Cauchy sequence

A sequence $\{v_n\} \subset V$ is a **Cauchy sequence** if

$$\boxed{\forall \varepsilon > 0,\ \exists N \in \mathbb{N} : m, n \geq N \Longrightarrow \|v_m - v_n\| < \varepsilon.}$$

Convergence compares v_n with a known candidate v. The Cauchy condition compares late terms with one another and does not require the limit to be known in advance. Every convergent sequence is Cauchy, since

$$\|v_m - v_n\| \leq \|v_m - v\| + \|v_n - v\|.$$

The converse depends on the space.

Complete space and Hilbert space

A normed space is **complete** if every Cauchy sequence converges in its norm to an element of the space. An inner-product space $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ is a **Hilbert space** if it is complete in the induced norm

$$\|f\| = \sqrt{\langle f, f \rangle}.$$

Every finite-dimensional normed space over $\mathbb{R}$ or $\mathbb{C}$ is complete. Completeness becomes a genuinely new issue in infinite dimensions. The inclusion $\mathbb{Q} \subset \mathbb{R}$ is a useful analogy: a rational Cauchy sequence may approach $\sqrt{2}$, which is missing from $\mathbb{Q}$.

4 A noncomplete function space

On $C([0, 1])$, define

$$\langle f, g \rangle = \int_0^1 \overline{f(x)} g(x)\, dx, \qquad \|f\|_2 = \left(\int_0^1 |f(x)|^2\, dx \right)^{1/2}.$$

This is a valid inner product and norm on continuous functions, but the space is not complete in this norm.

Example 6.1: continuous functions with a discontinuous L^2 limit

For $n \geq 2$, define

$$f_n(x) = \begin{cases} 0, & 0 \leq x \leq \frac{1}{2} - \frac{1}{n}, \\ n\left(x - \frac{1}{2} + \frac{1}{n}\right), & \frac{1}{2} - \frac{1}{n} < x < \frac{1}{2}, \\ 1, & \frac{1}{2} \leq x \leq 1. \end{cases}$$

Every f_n is continuous. The natural limit is the step function

$$H(x) = \begin{cases} 0, & x < 1/2, \\ 1, & x \geq 1/2, \end{cases}$$

which is not continuous at $x = 1/2$. The functions differ only on the transition interval, where $H = 0$. With $y = x - \frac{1}{2} + \frac{1}{n}$,

$$\|f_n - H\|_2^2 = n^2 \int_0^{1/n} y^2 dy = \frac{1}{3n}.$$

Therefore $f_n \to H$ in the L^2 norm. The sequence is Cauchy in $C([0,1])$ equipped with that norm, but its limit is not continuous. Hence $C([0,1])$ is not complete in the L^2 norm.

The failure lies in the space, not in the sequence: continuous functions are too few to contain every meaningful L^2 limit.

5 L^1, L^2, and improper power integrals

The spaces $L^1([a,b])$ and $L^2([a,b])$

For measurable functions $f : [a,b] \to \mathbb{K}$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$,

$$L^1([a,b]) = \left\{ f : \int_a^b |f(x)|\, dx < \infty \right\},$$

and

$$L^2([a,b]) = \left\{ f : \int_a^b |f(x)|^2\, dx < \infty \right\}.$$

The associated norms are

$$\|f\|_1 = \int_a^b |f(x)|\, dx, \qquad \|f\|_2 = \left(\int_a^b |f(x)|^2\, dx \right)^{1/2}.$$

The space $L^2([a,b])$ has the inner product

$$\boxed{\langle f, g \rangle = \int_a^b \overline{f(x)} g(x)\, dx},$$

and is complete. Thus it is a Hilbert space. The L^1 norm is also complete, but it does not arise from an inner product and therefore does not provide the same orthogonal geometry.

On a finite interval, every square-integrable function is absolutely integrable. Indeed, Cauchy–

Schwarz gives

$$\begin{aligned}\|f\|_1 &= \int_a^b |f(x)| \cdot 1 \, dx \\ &\le \left(\int_a^b |f(x)|^2 dx \right)^{1/2} \left(\int_a^b 1^2 dx \right)^{1/2} \\ &= \sqrt{b-a} \, \|f\|_2 \, .\end{aligned}$$

Consequently,

$$\boxed{L^2([a,b]) \subset L^1([a,b])} \qquad \text{when } b - a < \infty.$$

The reverse inclusion is false.

The model calculation near a singular endpoint is

$$\int_0^1 x^p dx = \lim_{\varepsilon \to 0^+} \int_\varepsilon^1 x^p dx.$$

If $p \neq -1$, then

$$\int_\varepsilon^1 x^p dx = \frac{1 - \varepsilon^{p+1}}{p+1}.$$

This has a finite limit exactly when $p + 1 > 0$. For $p = -1$, the integral is $-\log \varepsilon$, which diverges. Therefore,

$$\boxed{\int_0^1 x^p dx < \infty \quad \Longleftrightarrow \quad p > -1.}$$

Example 6.2: classifying a power singularity

Let $h_\beta(x) = x^{-\beta}$ on $(0, 1)$. Then

$$\|h_\beta\|_1 = \int_0^1 x^{-\beta} dx$$

is finite exactly when $-\beta > -1$, or $\beta < 1$. Likewise,

$$\|h_\beta\|_2^2 = \int_0^1 x^{-2\beta} dx$$

is finite exactly when $-2\beta > -1$, or $\beta < 1/2$. When $\beta < 1/2$,

$$\boxed{\|h_\beta\|_2 = \frac{1}{\sqrt{1 - 2\beta}}.}$$

For example, $x^{-3/4}$ belongs to $L^1((0,1))$ but not to $L^2((0,1))$.

A reliable membership test

To decide whether a function belongs to L^1 or L^2:

1. locate every endpoint or interior point where the function may be singular;
2. write the relevant integral as an improper integral;
3. test $|f|$ for L^1 and $|f|^2$ for L^2;
4. only after convergence is established, evaluate the norm.

6 Equality almost everywhere and indicator functions

Elements of L^1 and L^2 are equivalence classes of functions rather than pointwise-defined functions.

Equality almost everywhere

Functions f and g are equal **almost everywhere**, abbreviated a.e., if $f(x) = g(x)$ except possibly on a set of measure zero. Functions equal almost everywhere represent the same element of L^1 and L^2.

For example, changing the value of a function at one point does not change any of its integrals or its L^p equivalence class. This resolves the apparent problem that a nonzero pointwise function could otherwise have norm zero.

Indicator function

For a measurable set E, the indicator function of E is

$$\mathbf{1}_E(x) = \begin{cases} 1, & x \in E, \\ 0, & x \notin E. \end{cases}$$

If $\mu(E)$ denotes the length, or measure, of E, then

$$\|\mathbf{1}_E\|_1 = \mu(E), \qquad \|\mathbf{1}_E\|_2 = \sqrt{\mu(E)}.$$

For measurable sets A and B,

$$\langle \mathbf{1}_A, \mathbf{1}_B \rangle = \mu(A \cap B),$$

and, because $|\mathbf{1}_A - \mathbf{1}_B|^2$ is the indicator of the symmetric difference $A \triangle B$,

$$\boxed{\|\mathbf{1}_A - \mathbf{1}_B\|_2^2 = \mu(A \triangle B).}$$

More generally, if $B \subset A$ and $r, s \in \mathbb{C}$, then the sets B and $A \setminus B$ are disjoint, so

$$\boxed{\|r\mathbf{1}_A - s\mathbf{1}_B\|_2^2 = |r - s|^2 \mu(B) + |r|^2 \mu(A \setminus B).}$$

This formula is especially useful for comparing concentrated functions with nested supports.

Example 6.3: distance between two indicator functions

On $[0, 1]$, let $A = [0, 1/2]$ and $B = [1/3, 3/4]$. Their symmetric difference has length

$$\mu(A \triangle B) = \frac{1}{3} + \frac{1}{4} = \frac{7}{12}.$$

Therefore,

$$\|\mathbf{1}_A - \mathbf{1}_B\|_2 = \sqrt{\frac{7}{12}}.$$

The overlap $A \cap B = [1/3, 1/2]$ contributes nothing to the distance because the two functions agree there.

7 The sequence space ℓ^2 and tails of series

The Hilbert space ℓ^2

$$\ell^2 = \left\{ a = (a_1, a_2, \ldots) : \sum_{k=1}^{\infty} |a_k|^2 < \infty \right\}.$$

The inner product and norm are

$$\langle a, b \rangle = \sum_{k=1}^{\infty} \overline{a_k} b_k, \qquad \|a\|_{\ell^2} = \left(\sum_{k=1}^{\infty} |a_k|^2 \right)^{1/2}.$$

The space ℓ^2 is complete.

For the canonical vectors $e_1 = (1, 0, 0, \ldots)$, $e_2 = (0, 1, 0, \ldots)$, and so on, $\langle e_j, e_k \rangle = \delta_{jk}$. If $a = (a_1, a_2, \ldots) \in \ell^2$, then

$$a = \sum_{k=1}^{\infty} a_k e_k, \qquad \|a\|_{\ell^2}^2 = \sum_{k=1}^{\infty} |a_k|^2.$$

Suppose $a^{(N)} = (a_1, \ldots, a_N, 0, 0, \ldots)$ is the Nth truncation of a. If $M > N$, then

$$a^{(M)} - a^{(N)} = (0, \ldots, 0, a_{N+1}, \ldots, a_M, 0, \ldots),$$

and hence

$$\boxed{\left\| a^{(M)} - a^{(N)} \right\|_{\ell^2}^2 = \sum_{k=N+1}^{M} |a_k|^2.}$$

Because a convergent series has tails that tend to zero, $\sum_k |a_k|^2 < \infty$ implies that the truncations are Cauchy. Moreover,

$$\boxed{\left\| a - a^{(N)} \right\|_{\ell^2}^2 = \sum_{k=N+1}^{\infty} |a_k|^2 \longrightarrow 0.}$$

Example 6.4: a Cauchy sequence of truncations

Define

$$b^{(N)} = \left(\frac{1}{2}, \frac{1}{2^2}, \ldots, \frac{1}{2^N}, 0, 0, \ldots \right).$$

For $M > N$,

$$\begin{aligned} \left\| b^{(M)} - b^{(N)} \right\|_{\ell^2}^2 &= \sum_{k=N+1}^{M} \frac{1}{4^k} \\ &\leq \sum_{k=N+1}^{\infty} \frac{1}{4^k} = \frac{1}{3 \cdot 4^N}. \end{aligned}$$

Thus the sequence is Cauchy. Its natural limit is

$$b = \left(\frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \ldots \right),$$

and

$$\|b\|_{\ell^2}^2 = \sum_{k=1}^{\infty} \frac{1}{4^k} = \frac{1}{3}.$$

Finally,

$$\left\|b - b^{(N)}\right\|_{\ell^2}^2 = \frac{1}{3 \cdot 4^N} \to 0.$$

The tail estimate proves both the Cauchy property and convergence to the identified limit.

Where completeness enters

Tail estimates can prove that a sequence is Cauchy without first knowing its limit. Completeness then guarantees that some limit exists inside the space. When a natural coordinatewise candidate can also be identified, checking that the candidate lies in ℓ^2 and that the infinite tail tends to zero proves convergence directly.

8 Infinite orthonormal systems and bases

Orthonormal system and orthonormal basis

A sequence $\{u_k\}_{k=1}^{\infty}$ is orthonormal if

$$\langle u_j, u_k \rangle = \delta_{jk}.$$

It is an **orthonormal basis**, or complete orthonormal system, if every $f \in \mathcal{H}$ satisfies

$$\boxed{f = \sum_{k=1}^{\infty} \langle u_k, f \rangle \, u_k} \qquad \text{with convergence in norm.}$$

Equivalently,

$$\overline{\operatorname{span}\{u_1, u_2, \ldots\}} = \mathcal{H}.$$

Orthonormality alone does not guarantee completeness. The coefficients retain the finite-dimensional projection formula

$$\boxed{c_k = \langle u_k, f \rangle},$$

and the equality $f = \sum_k c_k u_k$ means precisely that

$$\|f - S_N\| \to 0, \qquad S_N = \sum_{k=1}^{N} c_k u_k.$$

9 Projection in L^2, Bessel, and best approximation

Let

$$S_N = \sum_{k=1}^{N} \langle u_k, f \rangle \, u_k, \qquad V_N = \operatorname{span}\{u_1, \ldots, u_N\}.$$

For $j \le N$,

$$\langle u_j, f - S_N \rangle = 0,$$

so the residual $f - S_N$ is orthogonal to V_N. The Pythagorean theorem gives

$$\boxed{\|f\|^2 = \sum_{k=1}^{N} |\langle u_k, f \rangle|^2 + \|f - S_N\|^2 .} \tag{1}$$

If $g \in V_N$, then

$$f - g = (f - S_N) + (S_N - g),$$

and the two terms are orthogonal. Hence

$$\|f - g\|^2 = \|f - S_N\|^2 + \|S_N - g\|^2 \geq \|f - S_N\|^2 .$$

Thus S_N is the unique best approximation to f from V_N.

Example 6.5: projection of a function onto linear polynomials

In $L^2([0,1])$, consider

$$u_0(x) = 1, \qquad u_1(x) = \sqrt{12}\left(x - \frac{1}{2}\right).$$

Direct integration shows that $\|u_0\|_2 = \|u_1\|_2 = 1$ and $\langle u_0, u_1 \rangle = 0$. For $f(x) = x^2$,

$$c_0 = \langle u_0, f \rangle = \int_0^1 x^2 dx = \frac{1}{3},$$

and

$$\begin{aligned} c_1 &= \sqrt{12} \int_0^1 \left(x - \frac{1}{2}\right) x^2 dx \\ &= \sqrt{12}\left(\frac{1}{4} - \frac{1}{6}\right) = \frac{\sqrt{3}}{6}. \end{aligned}$$

Therefore,

$$S_1 = c_0 u_0 + c_1 u_1 = x - \frac{1}{6}.$$

This is the best L^2 approximation to x^2 among all functions $a + bx$. The finite Pythagorean identity can be checked numerically:

$$\|f\|_2^2 = \frac{1}{5}, \qquad |c_0|^2 + |c_1|^2 = \frac{7}{36}, \qquad \|f - S_1\|_2^2 = \frac{1}{180},$$

and indeed $1/5 = 7/36 + 1/180$.

Since the final term in (1) is nonnegative, taking $N \to \infty$ gives Bessel's inequality.

Bessel's inequality and Parseval's identity

For any orthonormal system and every $f \in \mathcal{H}$,

$$\boxed{\sum_{k=1}^{\infty} |\langle u_k, f \rangle|^2 \leq \|f\|^2 .}$$

If the orthonormal system is complete, then $S_N \to f$, and

$$\boxed{\|f\|^2 = \sum_{k=1}^{\infty} |\langle u_k, f \rangle|^2.}$$

The latter equality is Parseval's identity.

Bessel shows that Fourier-type coefficients form an ℓ^2 sequence. Parseval says that a complete orthonormal basis captures all of the squared norm. If only finitely many coefficients are nonzero, Parseval reduces to an ordinary finite-dimensional Pythagorean identity.

10 Complex exponentials and Fourier coefficients

On $[0,1]$, define

$$u_k(x) = e^{2\pi ikx}, \qquad k \in \mathbb{Z}.$$

Then

$$\langle u_j, u_k \rangle = \int_0^1 e^{2\pi i(k-j)x} dx = \delta_{jk}.$$

The family $\{e^{2\pi ikx}\}_{k\in\mathbb{Z}}$ is a complete orthonormal system in $L^2([0,1])$. Therefore,

$$f(x) = \sum_{k\in\mathbb{Z}} c_k e^{2\pi ikx} \quad \text{in } L^2,$$

where

$$\boxed{c_k = \int_0^1 e^{-2\pi ikx} f(x) dx.}$$

Parseval becomes

$$\boxed{\int_0^1 |f(x)|^2 dx = \sum_{k\in\mathbb{Z}} |c_k|^2.}$$

The basic orthogonality integral is

$$\int_0^1 e^{2\pi irx} dx = \begin{cases} 1, & r = 0, \\ 0, & r \in \mathbb{Z} \setminus \{0\}. \end{cases}$$

It allows each Fourier coefficient to select exactly the matching mode.

Example 6.6: a function with finitely many Fourier modes

Let

$$h(x) = 3 - 2e^{2\pi ix} + ie^{-6\pi ix}.$$

The nonzero coefficients are

$$c_0 = 3, \qquad c_1 = -2, \qquad c_{-3} = i.$$

All other coefficients vanish. Parseval immediately gives

$$\|h\|_2^2 = |3|^2 + |-2|^2 + |i|^2 = 14.$$

For a direct check, expand $|h|^2 = \overline{h}h$. The three squared-amplitude terms integrate to $9+4+1$, while every cross term is a nonzero integer exponential and integrates to zero. Thus the direct integral is also 14.

Example 6.7: Fourier coefficients of $f(x) = x$

For $k = 0$, $c_0 = 1/2$. For $k \neq 0$, integration by parts gives

$$\begin{aligned} c_k &= \int_0^1 x e^{-2\pi ikx} dx \\ &= \left[\frac{x e^{-2\pi ikx}}{-2\pi ik} \right]_0^1 - \int_0^1 \frac{e^{-2\pi ikx}}{-2\pi ik} dx \\ &= -\frac{1}{2\pi ik} = \frac{i}{2\pi k}. \end{aligned}$$

Thus, in $L^2([0,1])$,

$$x = \frac{1}{2} + \sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{i}{2\pi k} e^{2\pi i k x}.$$

Parseval checks the result:

$$\int_0^1 x^2 dx = \frac{1}{4} + 2\sum_{k=1}^{\infty} \frac{1}{4\pi^2 k^2} = \frac{1}{3}.$$

11 Pointwise convergence versus L^2 convergence

Two different modes of convergence

The sequence f_n converges **pointwise** to f if, for every fixed x,

$$f_n(x) \to f(x).$$

It converges to f in L^2 if

$$\boxed{\|f_n - f\|_2 = \left(\int |f_n(x) - f(x)|^2 dx\right)^{1/2} \to 0.}$$

Pointwise convergence examines each point separately; L^2 convergence examines integrated squared error. Neither condition, by itself, is the same as the other.

Example 6.8: L^2 convergence with an exceptional point

Let $f_n(x) = x^n$ on $[0,1]$. Pointwise, $f_n(x) \to 0$ for $x < 1$, while $f_n(1) = 1$. Thus the pointwise limit has value 1 at the endpoint. In L^2, however,

$$\|f_n\|_2^2 = \int_0^1 x^{2n} dx = \frac{1}{2n+1} \to 0.$$

Hence $f_n \to 0$ in L^2. The value at the single endpoint is invisible to the L^2 norm.

Example 6.9: pointwise convergence without L^2 convergence

For $n \geq 1$, define

$$q_n(x) = \sqrt{n}\, \mathbf{1}_{[1/n,\, 2/n]}(x) \qquad \text{on } [0,1],$$

with the interval intersected with $[0,1]$ when $n = 1$. For every fixed $x \in [0,1]$, eventually x lies outside $[1/n, 2/n]$, so $q_n(x) \to 0$. However, for $n \geq 2$,

$$\|q_n\|_2^2 = n\left(\frac{2}{n} - \frac{1}{n}\right) = 1.$$

Therefore q_n does not converge to zero in L^2.
The failure is even visible in the dyadic subsequence. The supports of q_{2^j} and q_{2^j+1} overlap only at one endpoint, a measure-zero set. Consequently,

$$\|q_{2^j} - q_{2^j+1}\|_2^2 = \|q_{2^j}\|_2^2 + \|q_{2^j+1}\|_2^2 = 2.$$

This subsequence is not Cauchy, so the original sequence is not Cauchy in L^2.

The lesson is that a high, narrow concentration may disappear at each fixed point while retaining nonzero integrated energy. To test L^2 convergence, one must calculate a norm; pointwise inspection is not enough.

12 Cauchy partial sums and the role of completeness

Consider a formal orthonormal series

$$\sum_{k=1}^{\infty} c_k u_k, \qquad S_N = \sum_{k=1}^{N} c_k u_k.$$

For $m > n$, orthonormality gives

$$\boxed{\|S_m - S_n\|^2 = \sum_{k=n+1}^{m} |c_k|^2.}$$

If $\sum_k |c_k|^2 < \infty$, the tails tend to zero and $\{S_N\}$ is Cauchy. Completeness then guarantees a limit $f \in \mathcal{H}$.

Why completeness is essential

square-summable coefficients $\Longrightarrow$ Cauchy partial sums
$\Longrightarrow$ completeness $\Longrightarrow$ a limit inside $\mathcal{H}$

Without completeness, the partial sums could be Cauchy while pointing toward an object outside the space.

An orthonormal Hilbert basis should not be confused with a Hamel basis. A Hamel basis represents each vector by a finite linear combination. A Hilbert basis permits norm-convergent infinite expansions.

13 Conceptual summary

Concept	Meaning
Cauchy sequence	Late terms become arbitrarily close to one another.
Completeness	Every Cauchy sequence has a limit inside the space.
Hilbert space	A complete inner-product space.
L^1	Absolute integrability: $\int \lvert f\rvert < \infty$.
L^2	Square integrability: $\int \lvert f\rvert^2 < \infty$.
Equality a.e.	Changes on measure-zero sets do not change the L^p element.
Indicator function	Represents a measurable set as a function taking values 0 and 1.
Orthonormal basis	A complete orthonormal system with norm-convergent expansions.
Bessel	An orthonormal system captures no more than the total squared norm.
Parseval	A complete system captures exactly the total squared norm.
L^2 convergence	Integrated squared error tends to zero.

Common pitfalls

- A Cauchy sequence need not converge unless the space is complete.
- Testing L^2 membership requires integrating $\lvert f\rvert^2$, not $\lvert f\rvert$.
- On a finite interval, $L^2 \subset L^1$, but L^1 is not contained in L^2.
- L^2 elements are equivalence classes, not distinguished by values on measure-zero sets.
- Orthonormality does not imply completeness of the system.
- Bessel's inequality holds for any orthonormal system; Parseval's identity requires an orthonormal basis.
- Convergence in L^2 does not automatically imply pointwise or uniform convergence, and pointwise convergence does not imply L^2 convergence.

Core formulas

$$\|f\|_1 = \int_a^b |f(x)|dx, \qquad \|f\|_2^2 = \int_a^b |f(x)|^2 dx.$$

$$\int_0^1 x^p dx < \infty \iff p > -1, \qquad \|f\|_1 \le \sqrt{b-a}\, \|f\|_2 .$$

$$\left\| a^{(M)} - a^{(N)} \right\|_{\ell^2}^2 = \sum_{k=N+1}^{M} |a_k|^2.$$

$$c_k = \langle u_k, f \rangle , \qquad S_N = \sum_{k=1}^{N} \langle u_k, f \rangle \, u_k.$$

$$\|f\|^2 = \sum_{k=1}^{N} |\langle u_k, f\rangle|^2 + \|f - S_N\|^2 .$$

$$\sum_{k=1}^{\infty} |\langle u_k, f\rangle|^2 \le \|f\|^2 \quad \text{(Bessel)}, \qquad \|f\|^2 = \sum_{k=1}^{\infty} |\langle u_k, f\rangle|^2 \quad \text{(Parseval)}.$$

14 Take-home assignment: five problems

Special Thursday assignment — due Thursday, October 15, 2026

Complete all five problems and submit one organized solution set at the beginning of class on **Thursday, October 15, 2026**. Show intermediate steps, state the results you use, and explain your reasoning in complete sentences when requested. Answers without supporting work are incomplete.
No solutions are included in these notes.

Problem 1 — Cauchy sequences and completeness in ℓ^2

For each $n \in \mathbb{N}$, define

$$a^{(n)} = \left(1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, 0, 0, \ldots\right).$$

(a) For $m > n$, write $a^{(m)} - a^{(n)}$ explicitly.

(b) Show that

$$\left\|a^{(m)} - a^{(n)}\right\|_{\ell^2}^2 = \sum_{k=n+1}^{m} \frac{1}{k^2}.$$

(c) Use convergence of $\sum_{k=1}^{\infty} k^{-2}$ to prove that $\{a^{(n)}\}$ is Cauchy.

(d) Identify its natural limit a and prove that $a \in \ell^2$.

(e) Prove directly that $a^{(n)} \to a$ in ℓ^2 and explain where completeness enters conceptually.

Problem 2 — Membership in L^2

For real α, define $f_\alpha(x) = x^{-\alpha}$ on $(0, 1)$.

(a) Determine all α for which $f_\alpha \in L^1((0,1))$.

(b) Determine all α for which $f_\alpha \in L^2((0,1))$.

(c) Give an explicit value of α for which $f_\alpha \in L^1$ but $f_\alpha \notin L^2$.

(d) For every admissible α, compute $\|f_\alpha\|_2$.

(e) Explain why the value assigned to $f_\alpha(0)$ does not affect its element of L^2.

Problem 3 — Projection, Bessel, and best approximation

In $L^2([-1, 1])$, use the orthonormal functions

$$u_0(x) = \frac{1}{\sqrt{2}}, \qquad u_1(x) = \sqrt{\frac{3}{2}}\, x.$$

Let $f(x) = x^2 + x$.

(a) Verify that u_0 and u_1 are orthonormal.

(b) Compute $c_0 = \langle u_0, f\rangle$ and $c_1 = \langle u_1, f\rangle$.

(c) Construct the projection $S_1 = c_0 u_0 + c_1 u_1$.

(d) Compute $\|f\|^2$, $|c_0|^2 + |c_1|^2$, and $\|f - S_1\|^2$, and verify the finite Pythagorean identity.

(e) Explain why S_1 is the best approximation to f among all linear functions $a + bx$.

Problem 4 — Fourier coefficients and Parseval

Let $f(x) = e^{2\pi i x} + 2e^{-4\pi i x}$ on $[0, 1]$.

(a) Compute every Fourier coefficient

$$c_k = \int_0^1 e^{-2\pi i k x} f(x) dx.$$

(b) Write the Fourier expansion of f and identify all nonzero modes.

(c) Compute $\int_0^1 |f(x)|^2 dx$ directly.

(d) Compute $\sum_{k \in \mathbb{Z}} |c_k|^2$ and verify Parseval's identity.

(e) Interpret the result as preservation of squared norm between the physical-space and coefficient-space representations.

Problem 5 — Comparing modes of convergence

On $[0, 1]$, define

$$g_n(x) = \sqrt{n}\, \mathbf{1}_{[0,1/n]}(x),$$

where $\mathbf{1}_E$ is the indicator function of the set E.

(a) Determine the pointwise limit of $g_n(x)$ for every $x \in [0, 1]$.

(b) Compute $\|g_n\|_2$.

(c) Decide whether $g_n \to 0$ in $L^2([0, 1])$ and justify the answer.

(d) Compute $\|g_n - g_m\|_2^2$ for $m > n$ and determine whether $\{g_n\}$ is a Cauchy sequence in L^2.

(e) Explain clearly what this example shows about the difference between pointwise convergence and L^2 convergence.

End of Lecture 06 student notes