GES-207-4 — Mathematical Methods in Physics

Lecture 05

Hermitian and Unitary Operators and the Spectral Theorem

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Learning goals

This lecture extends the geometry and diagonalization developed in the previous lectures from real vector spaces to complex inner-product spaces. By the end of these notes, you should be able to:

1. explain why complex inner products require conjugation;
2. define and compute the adjoint of an operator;
3. recognize Hermitian, anti-Hermitian, and unitary operators;
4. prove the fundamental spectral properties of these classes;
5. apply the Spectral Theorem and construct a unitary diagonalization;
6. handle degenerate eigenspaces using Gram–Schmidt;
7. construct spectral projectors and functions of operators;
8. connect these ideas with quantum mechanics, principal axes, and anisotropic transport.

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1 From real symmetry to complex inner products

In Lecture 3, the inner product allowed us to define lengths, angles, orthogonality, projections, and orthonormal bases. In Lecture 4, eigenvalues, eigenvectors, and diagonalization gave us a way to identify the natural modes of a linear operator. This lecture brings those two lines of thought together.

For a real symmetric matrix,

$$A^T = A.$$

Such a matrix has real eigenvalues and can be diagonalized using an orthogonal matrix:

$$A = QDQ^T, \qquad Q^TQ = I.$$

Many physical problems, however, naturally involve complex vectors. Examples include oscillations represented by complex exponentials, Fourier modes, wave amplitudes, and quantum states. In $\mathbb{C}^n$, the ordinary transpose is not enough because the inner product itself contains complex conjugation.

Throughout these notes we use the convention

$$\boxed{\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u}^\dagger \mathbf{v} = \sum_{j=1}^{n} \overline{u_j} v_j.} \tag{1}$$

Thus, the inner product is conjugate-linear in its first argument and linear in its second argument:

$$\langle \alpha\mathbf{u} + \beta\mathbf{w}, \mathbf{v} \rangle = \overline{\alpha} \langle \mathbf{u}, \mathbf{v} \rangle + \overline{\beta} \langle \mathbf{w}, \mathbf{v} \rangle , \tag{2}$$

$$\langle \mathbf{u}, \alpha\mathbf{v} + \beta\mathbf{w} \rangle = \alpha \langle \mathbf{u}, \mathbf{v} \rangle + \beta \langle \mathbf{u}, \mathbf{w} \rangle . \tag{3}$$

It also satisfies

$$\langle \mathbf{u}, \mathbf{v} \rangle = \overline{\langle \mathbf{v}, \mathbf{u} \rangle}, \qquad \langle \mathbf{v}, \mathbf{v} \rangle > 0 \quad \text{when } \mathbf{v} \neq \mathbf{0}.$$

Why is conjugation necessary? Take

$$\mathbf{v} = \begin{pmatrix} 1 \\ i \end{pmatrix}.$$

Without conjugation,

$$\mathbf{v}^T\mathbf{v} = 1 + i^2 = 0,$$

even though $\mathbf{v}$ is not the zero vector. With the correct complex inner product,

$$\mathbf{v}^\dagger\mathbf{v} = \begin{pmatrix} 1 & -i \end{pmatrix} \begin{pmatrix} 1 \\ i \end{pmatrix} = 2 > 0.$$

The conjugation that makes lengths positive is also what determines the correct complex analogue of matrix symmetry.

Real and complex cases side by side

Real space	Complex space
$\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u}^T\mathbf{v}$	$\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u}^\dagger\mathbf{v}$
$A^T = A$	$A^\dagger = A$
$Q^TQ = I$	$U^\dagger U = I$
$A = QDQ^T$	$A = UDU^\dagger$

For real matrices, conjugation changes nothing, so $A^\dagger = A^T$. Hermitian matrices are therefore the natural complex generalization of real symmetric matrices.

2 The adjoint: moving an operator through an inner product

Before stating the definition, consider a matrix $A \in \mathbb{C}^{n\times n}$ acting on the second vector:

$$\langle \mathbf{u}, A\mathbf{v}\rangle = \mathbf{u}^\dagger A\mathbf{v}.$$

Suppose we want an operator B acting on the first vector such that

$$\langle B\mathbf{u}, \mathbf{v}\rangle = \langle \mathbf{u}, A\mathbf{v}\rangle$$

for every $\mathbf{u}$ and $\mathbf{v}$. The left-hand side is

$$\langle B\mathbf{u}, \mathbf{v}\rangle = (B\mathbf{u})^\dagger \mathbf{v} = \mathbf{u}^\dagger B^\dagger \mathbf{v}.$$

Equality for all vectors requires $B^\dagger = A$, or $B = A^\dagger$. Therefore, when a matrix moves from the second slot to the first slot of the inner product, it becomes its conjugate transpose.

Definition: adjoint operator

Let V be a finite-dimensional inner-product space and let $T : V \to V$ be linear. The **adjoint** of T is the unique operator $T^\dagger$ such that

$$\boxed{\left\langle T^\dagger \mathbf{u}, \mathbf{v}\right\rangle = \langle \mathbf{u}, T\mathbf{v}\rangle} \qquad \text{for all } \mathbf{u}, \mathbf{v} \in V. \tag{4}$$

In an orthonormal basis of $\mathbb{C}^n$,

$$\boxed{A^\dagger = \overline{A}^{\,T}}, \qquad (A^\dagger)_{jk} = \overline{A_{kj}}.$$

The phrase "move the operator through the inner product" is useful, but it must be understood precisely:

$$\langle \mathbf{u}, T\mathbf{v}\rangle = \left\langle T^\dagger \mathbf{u}, \mathbf{v}\right\rangle.$$

The operator does not merely change position; it changes into its adjoint.

Example 5.1: computing a conjugate transpose

Let

$$A = \begin{pmatrix} 2 & 1+i \\ 3i & 4 \end{pmatrix}.$$

First conjugate every entry:

$$\overline{A} = \begin{pmatrix} 2 & 1-i \\ -3i & 4 \end{pmatrix}.$$

Then transpose:

$$\boxed{A^\dagger = \begin{pmatrix} 2 & -3i \\ 1-i & 4 \end{pmatrix}.}$$

For instance, $A_{21} = 3i$, so $(A^\dagger)_{12} = \overline{A_{21}} = -3i$.

For compatible matrices A, B and scalars $\alpha, \beta \in \mathbb{C}$,

$$(A+B)^\dagger = A^\dagger + B^\dagger, \tag{5}$$
$$(\alpha A)^\dagger = \overline{\alpha} A^\dagger, \tag{6}$$
$$(AB)^\dagger = B^\dagger A^\dagger, \tag{7}$$
$$(A^\dagger)^\dagger = A. \tag{8}$$

The reversed order in (7) follows from

$$(AB)^\dagger = \overline{AB}^{\,T} = (\overline{A}\,\overline{B})^T = \overline{B}^{\,T}\overline{A}^{\,T} = B^\dagger A^\dagger.$$

Thus $(ABC)^\dagger = C^\dagger B^\dagger A^\dagger$.

3 Hermitian, anti-Hermitian, and unitary operators

Three fundamental classes

An operator is **Hermitian**, or self-adjoint, if

$$\boxed{T^\dagger = T}.$$

It is **anti-Hermitian** if

$$\boxed{T^\dagger = -T}.$$

It is **unitary** if

$$\boxed{U^\dagger U = UU^\dagger = I}.$$

Consequently, a unitary operator satisfies

$$\boxed{U^{-1} = U^\dagger}.$$

For

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad A^\dagger = \begin{pmatrix} \overline{a} & \overline{c} \\ \overline{b} & \overline{d} \end{pmatrix},$$

the condition $A^\dagger = A$ implies

$$a = \overline{a}, \qquad d = \overline{d}, \qquad b = \overline{c}.$$

Hence every 2×2 Hermitian matrix has the form

$$\boxed{A = \begin{pmatrix} a & z \\ \overline{z} & d \end{pmatrix}, \qquad a, d \in \mathbb{R}, quadz \in \mathbb{C}.}$$

In particular, the diagonal entries are real.

If $K^\dagger = -K$, its diagonal entries satisfy $\overline{K_{jj}} = -K_{jj}$ and are purely imaginary or zero. If H is Hermitian, then

$$(iH)^\dagger = -iH,$$

so iH is anti-Hermitian.

Example 5.2: Hermitian does not mean simply symmetric

Consider

$$A = \begin{pmatrix} 2 & i \\ -i & 2 \end{pmatrix}.$$

Then

$$A^\dagger = \begin{pmatrix} 2 & i \\ -i & 2 \end{pmatrix} = A,$$

so A is Hermitian. However,

$$A^T = \begin{pmatrix} 2 & -i \\ i & 2 \end{pmatrix} \neq A.$$

This is why the ordinary transpose must be replaced by the conjugate transpose in complex spaces.

Example 5.3: an anti-Hermitian matrix

For

$$K = \begin{pmatrix} i & 2 \\ -2 & -i \end{pmatrix},$$

we obtain

$$K^\dagger = \begin{pmatrix} -i & -2 \\ 2 & i \end{pmatrix} = -K.$$

Therefore, K is anti-Hermitian.

4 Unitary operators preserve geometry

If U is unitary, then

$$\begin{aligned} \langle U\mathbf{u}, U\mathbf{v}\rangle &= (U\mathbf{u})^\dagger (U\mathbf{v}) \\ &= \mathbf{u}^\dagger U^\dagger U\mathbf{v} \\ &= \mathbf{u}^\dagger \mathbf{v} \\ &= \langle \mathbf{u}, \mathbf{v}\rangle . \end{aligned}$$

Thus

$$\boxed{\langle U\mathbf{u}, U\mathbf{v}\rangle = \langle \mathbf{u}, \mathbf{v}\rangle .} \tag{9}$$

Setting $\mathbf{u} = \mathbf{v}$ gives

$$\boxed{\|U\mathbf{v}\| = \|\mathbf{v}\| .}$$

Unitary operators preserve norms, distances, angles, and orthogonality. They are the complex analogues of real orthogonal transformations.

Example 5.4: a complex unitary matrix

Let

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}.$$

Then

$$U^\dagger = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}$$

and

$$\begin{aligned} U^\dagger U &= \frac{1}{2} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = I. \end{aligned}$$

Hence U is unitary.

5 Fundamental spectral properties

5.1 Hermitian eigenvalues are real

Theorem: reality of Hermitian eigenvalues

If $T = T^\dagger$ and

$$T\mathbf{v} = \lambda\mathbf{v}, \qquad \mathbf{v} \neq \mathbf{0},$$

then

$$\boxed{\lambda \in \mathbb{R}.}$$

Proof. Using the eigenvalue equation in the second slot,

$$\begin{aligned}\langle \mathbf{v}, T\mathbf{v}\rangle &= \langle \mathbf{v}, \lambda\mathbf{v}\rangle \\ &= \lambda \langle \mathbf{v}, \mathbf{v}\rangle .\end{aligned}$$

The scalar leaves the second slot *without* conjugation because the inner product is linear in that slot.

Now move T through the inner product. Since $T^\dagger = T$,

$$\begin{aligned}\langle \mathbf{v}, T\mathbf{v}\rangle &= \left\langle T^\dagger\mathbf{v}, \mathbf{v}\right\rangle \\ &= \langle T\mathbf{v}, \mathbf{v}\rangle \\ &= \langle \lambda\mathbf{v}, \mathbf{v}\rangle \\ &= \overline{\lambda} \langle \mathbf{v}, \mathbf{v}\rangle .\end{aligned}$$

The conjugate appears in the last line because λ is now in the *first*, conjugate-linear slot. Comparing the two expressions,

$$(\lambda - \overline{\lambda}) \langle \mathbf{v}, \mathbf{v}\rangle = 0.$$

Since $\mathbf{v} \neq \mathbf{0}$, its squared norm is positive. Therefore, $\lambda = \overline{\lambda}$, which means $\lambda \in \mathbb{R}$. □

5.2 Eigenvectors associated with distinct eigenvalues are orthogonal

Theorem: orthogonality of Hermitian eigenvectors

Suppose $T = T^\dagger$ and

$$T\mathbf{v}_1 = \lambda_1\mathbf{v}_1, \qquad T\mathbf{v}_2 = \lambda_2\mathbf{v}_2, \qquad \lambda_1 \neq \lambda_2.$$

Then

$$\boxed{\langle \mathbf{v}_1, \mathbf{v}_2\rangle = 0.}$$

Proof. Acting on the second vector gives

$$\langle \mathbf{v}_1, T\mathbf{v}_2\rangle = \lambda_2 \langle \mathbf{v}_1, \mathbf{v}_2\rangle .$$

Moving the operator to the first slot gives

$$\begin{aligned}\langle \mathbf{v}_1, T\mathbf{v}_2\rangle &= \left\langle T^\dagger\mathbf{v}_1, \mathbf{v}_2\right\rangle \\ &= \langle T\mathbf{v}_1, \mathbf{v}_2\rangle \\ &= \langle \lambda_1\mathbf{v}_1, \mathbf{v}_2\rangle \\ &= \overline{\lambda_1} \langle \mathbf{v}_1, \mathbf{v}_2\rangle .\end{aligned}$$

The conjugate appears because λ_1 is in the first slot. Since a Hermitian eigenvalue is real, $\overline{\lambda_1} = \lambda_1$. Hence

$$(\lambda_2 - \lambda_1)\langle \mathbf{v}_1, \mathbf{v}_2 \rangle = 0.$$

The eigenvalues are distinct, so the inner product must vanish. □

Important note

Automatic orthogonality applies to eigenvectors belonging to **distinct eigenvalues**. Arbitrary vectors in the same degenerate eigenspace need not be orthogonal. Gram–Schmidt can be used inside that eigenspace.

5.3 Unitary and anti-Hermitian spectra

If $U\mathbf{v} = \lambda\mathbf{v}$ and U is unitary, norm preservation gives

$$\|\mathbf{v}\| = \|U\mathbf{v}\| = \|\lambda\mathbf{v}\| = |\lambda|\,\|\mathbf{v}\|\,.$$

Since $\mathbf{v} \neq \mathbf{0}$,

$$\boxed{|\lambda| = 1.}$$

Every unitary eigenvalue lies on the unit circle and may be written as $\lambda = e^{i\theta}$.

If $K^\dagger = -K$ and $K\mathbf{v} = \mu\mathbf{v}$, then

$$\langle \mathbf{v}, K\mathbf{v} \rangle = \mu\,\langle \mathbf{v}, \mathbf{v} \rangle\,.$$

On the other hand,

$$\begin{aligned} \langle \mathbf{v}, K\mathbf{v} \rangle &= \left\langle K^\dagger\mathbf{v}, \mathbf{v} \right\rangle \\ &= \langle -K\mathbf{v}, \mathbf{v} \rangle \\ &= -\overline{\mu}\,\langle \mathbf{v}, \mathbf{v} \rangle\,. \end{aligned}$$

Therefore $\mu = -\overline{\mu}$, so

$$\boxed{\mu \in i\mathbb{R}.}$$

Where the spectra live

$$\begin{aligned} H^\dagger = H &\quad\Longrightarrow\quad \sigma(H) \subset \mathbb{R}, \\ K^\dagger = -K &\quad\Longrightarrow\quad \sigma(K) \subset i\mathbb{R}, \\ U^\dagger U = I &\quad\Longrightarrow\quad \sigma(U) \subset \{z \in \mathbb{C} : |z| = 1\}. \end{aligned}$$

Here $\sigma(A)$ denotes the spectrum of A, which in finite dimensions is the set of its eigenvalues.

6 The Spectral Theorem

The preceding results tell us that Hermitian eigenvalues are real and that eigenspaces associated with distinct eigenvalues are orthogonal. The Spectral Theorem guarantees that there are enough eigenvectors to form an orthonormal basis of the entire space.

Spectral Theorem for Hermitian operators

Let $T : V \to V$ be Hermitian on a finite-dimensional complex inner-product space. Then V has an orthonormal basis

$$\{\mathbf{u}_1, \dots, \mathbf{u}_n\}$$

of eigenvectors of T, and the corresponding eigenvalues $\lambda_1, \dots, \lambda_n$ are real.
If

$$U = [\mathbf{u}_1 | \cdots | \mathbf{u}_n], \qquad D = \operatorname{diag}(\lambda_1, \dots, \lambda_n),$$

then

$$\boxed{U^\dagger U = I}, \qquad \boxed{U^\dagger A U = D}, \qquad \boxed{A = U D U^\dagger}.$$

Because the columns of U are orthonormal,

$$(U^\dagger U)_{jk} = \mathbf{u}_j^\dagger \mathbf{u}_k = \delta_{jk}.$$

Thus $U^{-1} = U^\dagger$. The eigenvalue equations collected column by column give

$$AU = UD.$$

Multiplying by $U^\dagger$ yields both $U^\dagger A U = D$ and $A = U D U^\dagger$.

General diagonalization versus unitary diagonalization

For a general diagonalizable matrix, Lecture 4 gave

$$A = P D P^{-1}.$$

The eigenvectors in P need not be orthogonal, and P^{-1} must generally be computed separately. For a Hermitian matrix,

$$A = U D U^\dagger, \qquad U^{-1} = U^\dagger,$$

where the columns of U form an orthonormal basis and all entries of D are real. This is a substantially stronger conclusion.

7 A complete worked example

Example 5.5: complete Hermitian diagonalization

Consider

$$A = \begin{pmatrix} 2 & i \\ -i & 2 \end{pmatrix}.$$

Step 1: verify Hermiticity.

$$\overline{A} = \begin{pmatrix} 2 & -i \\ i & 2 \end{pmatrix}, \qquad A^\dagger = \overline{A}^{\,T} = \begin{pmatrix} 2 & i \\ -i & 2 \end{pmatrix} = A.$$

Step 2: find the eigenvalues.

$$\begin{aligned} p_A(\lambda) &= \det(A - \lambda I) \\ &= \det \begin{pmatrix} 2-\lambda & i \\ -i & 2-\lambda \end{pmatrix} \\ &= (2-\lambda)^2 - i(-i) \\ &= (2-\lambda)^2 - 1 \\ &= (\lambda - 1)(\lambda - 3). \end{aligned}$$

Hence

$$\boxed{\lambda_1 = 1, \qquad \lambda_2 = 3.}$$

Step 3: find and normalize an eigenvector for $\lambda_1 = 1$.

$$(A - I)\mathbf{v} = \mathbf{0} \quad \Longrightarrow \quad \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{0}.$$

The first equation gives $x + iy = 0$, so $x = -iy$. Choosing $y = 1$,

$$\mathbf{v}_1 = \begin{pmatrix} -i \\ 1 \end{pmatrix}, \qquad \|\mathbf{v}_1\| = \sqrt{2},$$

and therefore

$$\boxed{\mathbf{u}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} -i \\ 1 \end{pmatrix}.}$$

A direct multiplication confirms $A\mathbf{u}_1 = \mathbf{u}_1$.

Step 4: find and normalize an eigenvector for $\lambda_2 = 3$.

$$(A - 3I)\mathbf{v} = \mathbf{0} \quad \Longrightarrow \quad \begin{pmatrix} -1 & i \\ -i & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{0}.$$

The first equation gives $x = iy$. Taking $y = 1$ and normalizing,

$$\boxed{\mathbf{u}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} i \\ 1 \end{pmatrix}.}$$

Direct multiplication gives $A\mathbf{u}_2 = 3\mathbf{u}_2$.

Step 5: verify orthogonality.

$$\begin{aligned} \langle \mathbf{u}_1, \mathbf{u}_2 \rangle &= \mathbf{u}_1^\dagger \mathbf{u}_2 \\ &= \frac{1}{2} \begin{pmatrix} i & 1 \end{pmatrix} \begin{pmatrix} i \\ 1 \end{pmatrix} \\ &= \frac{1}{2}(i^2 + 1) = 0. \end{aligned}$$

Step 6: construct U and D.

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} -i & i \\ 1 & 1 \end{pmatrix}, \qquad D = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}, \qquad U^\dagger = \frac{1}{\sqrt{2}} \begin{pmatrix} i & 1 \\ -i & 1 \end{pmatrix}.$$

Step 7: verify the factorization.

$$U^\dagger U = \frac{1}{2} \begin{pmatrix} i & 1 \\ -i & 1 \end{pmatrix} \begin{pmatrix} -i & i \\ 1 & 1 \end{pmatrix} = I.$$

Also, $AU = UD$, so $U^\dagger AU = D$. Finally,

$$\begin{aligned} UDU^\dagger &= \frac{1}{2}\begin{pmatrix} -i & 3i \\ 1 & 3 \end{pmatrix}\begin{pmatrix} i & 1 \\ -i & 1 \end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} 4 & 2i \\ -2i & 4 \end{pmatrix} \\ &= \begin{pmatrix} 2 & i \\ -i & 2 \end{pmatrix} = A. \end{aligned}$$

Every part of the Spectral Theorem has now been verified explicitly.

8 Degeneracy and Gram–Schmidt inside eigenspaces

An eigenvalue is **degenerate** when its eigenspace has dimension greater than one. For example,

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{pmatrix}$$

has the eigenvalue 2 with eigenspace

$$E_2 = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}.$$

If $\mathbf{v}_1, \mathbf{v}_2 \in E_\lambda$, then any linear combination $\mathbf{w} = a\mathbf{v}_1 + b\mathbf{v}_2$ also belongs to E_λ because

$$A\mathbf{w} = aA\mathbf{v}_1 + bA\mathbf{v}_2 = \lambda(a\mathbf{v}_1 + b\mathbf{v}_2) = \lambda\mathbf{w}.$$

Gram–Schmidt uses only linear combinations of its input vectors. Therefore, applying it to a basis of E_λ produces an orthonormal basis that remains inside E_λ. For two independent vectors,

$$\begin{aligned} \mathbf{w}_1 &= \mathbf{v}_1, \\ \mathbf{w}_2 &= \mathbf{v}_2 - \frac{\langle \mathbf{w}_1, \mathbf{v}_2 \rangle}{\langle \mathbf{w}_1, \mathbf{w}_1 \rangle}\mathbf{w}_1, \end{aligned}$$

followed by $\mathbf{u}_j = \mathbf{w}_j / \|\mathbf{w}_j\|$.

9 Spectral projectors and spectral decomposition

In Lecture 3, the projection of $\mathbf{v}$ onto a normalized vector $\mathbf{u}_j$ was

$$\operatorname{proj}_{\mathbf{u}_j}(\mathbf{v}) = \langle \mathbf{u}_j, \mathbf{v} \rangle \, \mathbf{u}_j.$$

Since $\langle \mathbf{u}_j, \mathbf{v} \rangle = \mathbf{u}_j^\dagger \mathbf{v}$,

$$\operatorname{proj}_{\mathbf{u}_j}(\mathbf{v}) = \mathbf{u}_j \mathbf{u}_j^\dagger \mathbf{v}.$$

Spectral projector and spectral decomposition

For a normalized eigenvector $\mathbf{u}_j$, define

$$\boxed{P_j = \mathbf{u}_j \mathbf{u}_j^\dagger.}$$

This is the orthogonal projector onto the direction of $\mathbf{u}_j$. If $A\mathbf{u}_j = \lambda_j \mathbf{u}_j$ and the eigenvectors form an orthonormal basis, then

$$\boxed{A = \sum_{j=1}^{n} \lambda_j P_j.}$$

This expression is the **spectral decomposition** of A.

Orthonormality gives

$$P_j^2 = P_j, \qquad P_j^\dagger = P_j, \qquad P_j P_k = 0 \quad (j \neq k), \qquad \sum_j P_j = I.$$

The operator first separates a vector into orthogonal components and then multiplies each component by its corresponding eigenvalue:

$$A\mathbf{v} = \sum_j \lambda_j P_j \mathbf{v}.$$

Example 5.6: spectral projectors

For the matrix in Example 5.5,

$$P_1 = \frac{1}{2}\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix},$$
$$P_2 = \frac{1}{2}\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}.$$

One checks that $P_1 + P_2 = I$ and $P_1 P_2 = 0$. Since the eigenvalues are 1 and 3,

$$\boxed{A = P_1 + 3P_2.}$$

For a degenerate eigenvalue λ, let $\{\mathbf{u}_1, \ldots, \mathbf{u}_r\}$ be an orthonormal basis of its eigenspace. The projector onto the full eigenspace is

$$P_\lambda = \sum_{j=1}^{r} \mathbf{u}_j \mathbf{u}_j^\dagger.$$

The decomposition may then be grouped over distinct eigenvalues:

$$\boxed{A = \sum_\lambda \lambda P_\lambda.}$$

Although the orthonormal basis inside a degenerate eigenspace is not unique, the projector P_λ is unique.

10 Functions of operators and the matrix exponential

Lecture 4 showed that diagonalization reduces functions of matrices to functions of scalars. If $A = UDU^\dagger$, then

$$f(A) = U f(D) U^\dagger,$$

where

$$f(D) = \operatorname{diag}\big(f(\lambda_1), \ldots, f(\lambda_n)\big).$$

Using the spectral projectors,

$$\boxed{f(A) = \sum_j f(\lambda_j) P_j.} \tag{10}$$

The projectors remain unchanged; the scalar function acts on the eigenvalues.

Indeed, because $P_j P_k = 0$ for $j \neq k$ and $P_j^2 = P_j$,

$$A^m = \sum_j \lambda_j^m P_j.$$

Substitution into the power series for the exponential gives

$$\boxed{e^{At} = \sum_j e^{\lambda_j t} P_j.}$$

Example 5.7: exponential from the spectral decomposition

Since $A = P_1 + 3P_2$,

$$e^{At} = e^t P_1 + e^{3t} P_2.$$

Therefore,

$$\boxed{e^{At} = \frac{1}{2}\begin{pmatrix} e^t + e^{3t} & i(e^{3t} - e^t) \\ -i(e^{3t} - e^t) & e^t + e^{3t} \end{pmatrix}.}$$

At $t = 0$, $e^{A0} = P_1 + P_2 = I$, and

$$\frac{d}{dt} e^{At}\Big|_{t=0} = P_1 + 3P_2 = A.$$

These checks agree with the defining properties of the matrix exponential.

11 Physical applications

11.1 Quantum observables and expectation values

In quantum mechanics, a normalized state is represented by $|\psi\rangle$, with $\langle\psi|\psi\rangle = 1$. An observable is represented by a Hermitian operator $\widehat{A}$. If

$$\widehat{A}|u_k\rangle = a_k|u_k\rangle,$$

then the possible values a_k are real. The Spectral Theorem gives

$$\widehat{A} = \sum_k a_k P_k,$$

where $P_k = |u_k\rangle\langle u_k|$ in the nondegenerate case.

Expanding

$$|\psi\rangle = \sum_k c_k |u_k\rangle, \qquad c_k = \langle u_k|\psi\rangle,$$

the normalization condition becomes $\sum_k |c_k|^2 = 1$, and the expectation value is

$$\boxed{\langle \widehat{A} \rangle_\psi = \langle \psi | \widehat{A} | \psi \rangle = \sum_k a_k |c_k|^2.}$$

This is real because it is a weighted sum of real eigenvalues. More generally, for every Hermitian matrix A,

$$\overline{\langle \mathbf{v}, A\mathbf{v} \rangle} = \langle A\mathbf{v}, \mathbf{v} \rangle = \left\langle \mathbf{v}, A^\dagger \mathbf{v} \right\rangle = \langle \mathbf{v}, A\mathbf{v} \rangle \,,$$

so $\langle \mathbf{v}, A\mathbf{v} \rangle \in \mathbb{R}$.

11.2 Unitary evolution generated by a Hermitian operator

If $H^\dagger = H$, then $K = -iH$ is anti-Hermitian because

$$K^\dagger = (-iH)^\dagger = iH = -K.$$

Define

$$U(t) = e^{-iHt}.$$

Then

$$U(t)^\dagger = e^{iHt},$$

and, since iHt and $-iHt$ commute,

$$U(t)^\dagger U(t) = e^{iHt} e^{-iHt} = I.$$

Thus

$$\boxed{H^\dagger = H \quad \Longrightarrow \quad e^{-iHt} \text{ is unitary.}}$$

Unitary time evolution preserves the norm of a quantum state.

11.3 Physical tensors and principal axes

Many physical tensors are represented in orthonormal bases by real symmetric matrices. They admit the spectral representation

$$A = \sum_j \lambda_j \mathbf{u}_j \mathbf{u}_j^T.$$

The eigenvectors are the **principal directions**, and the eigenvalues are the **principal values**. In this basis, the tensor is diagonal and does not mix different components. This structure appears in inertia, stress, strain, covariance, susceptibility, permittivity, and symmetric transport tensors.

11.4 Anisotropic transport in a magnetized plasma

Let

$$\mathbf{B} = B\widehat{\mathbf{b}}, \qquad \widehat{\mathbf{b}} = \frac{\mathbf{B}}{|\mathbf{B}|}.$$

The magnetic field selects a preferred direction. Define

$$P_\parallel = \widehat{\mathbf{b}}\widehat{\mathbf{b}}^T, \qquad P_\perp = I - P_\parallel.$$

Then

$$P_\parallel \mathbf{v} = (\widehat{\mathbf{b}} \cdot \mathbf{v})\widehat{\mathbf{b}},$$
$$P_\perp \mathbf{v} = \mathbf{v} - (\widehat{\mathbf{b}} \cdot \mathbf{v})\widehat{\mathbf{b}}.$$

Example 5.8: symmetric part of the conductivity tensor

For $\mathbf{J} = \boldsymbol{\sigma}\mathbf{E}$, a simple anisotropic model for the symmetric part is

$$\boxed{\boldsymbol{\sigma}_{\mathrm{s}} = \sigma_{\parallel} P_{\parallel} + \sigma_{\perp} P_{\perp}.}$$

Since

$$\boldsymbol{\sigma}_{\mathrm{s}} \widehat{\mathbf{b}} = \sigma_{\parallel} \widehat{\mathbf{b}},$$

the field direction is an eigenvector with eigenvalue $\sigma_{\parallel}$. For any $\mathbf{v}_{\perp}$ satisfying $\widehat{\mathbf{b}} \cdot \mathbf{v}_{\perp} = 0$,

$$\boldsymbol{\sigma}_{\mathrm{s}} \mathbf{v}_{\perp} = \sigma_{\perp} \mathbf{v}_{\perp}.$$

Thus $\sigma_{\perp}$ is degenerate, with the plane perpendicular to $\mathbf{B}$ as its eigenspace. The formula for $\boldsymbol{\sigma}_{\mathrm{s}}$ is already a spectral decomposition.

The Hall component must be treated separately

The full conductivity tensor of a magnetized plasma may also contain an antisymmetric Hall component. In a basis with $\widehat{\mathbf{b}} = \widehat{\mathbf{z}}$, a typical form is

$$\boldsymbol{\sigma} = \begin{pmatrix} \sigma_{\perp} & -\sigma_H & 0 \\ \sigma_H & \sigma_{\perp} & 0 \\ 0 & 0 & \sigma_{\parallel} \end{pmatrix}.$$

The real orthogonal decomposition using $P_{\parallel}$ and $P_{\perp}$ applies to the **symmetric part** of the response. It does not assert that the complete conductivity tensor is symmetric when $\sigma_H \neq 0$.

12 Practical workflow and summary

For a candidate Hermitian matrix A:

1. Compute $A^{\dagger} = \overline{A}^{T}$ and verify $A^{\dagger} = A$.
2. Find $p_A(\lambda) = \det(A - \lambda I)$ and its roots.
3. Check that all eigenvalues are real.
4. Solve $(A - \lambda I)\mathbf{v} = \mathbf{0}$ for each eigenvalue.
5. Normalize with $\|\mathbf{v}\| = \sqrt{\mathbf{v}^{\dagger}\mathbf{v}}$.
6. Use Gram–Schmidt inside any degenerate eigenspace if needed.
7. Build $U = [\mathbf{u}_1 | \cdots | \mathbf{u}_n]$ and verify $U^{\dagger}U = I$.
8. Build D in the same eigenvalue order as the columns of U.
9. Verify $AU = UD$, $U^{\dagger}AU = D$, or $A = UDU^{\dagger}$.
10. When useful, build $P_j = \mathbf{u}_j \mathbf{u}_j^{\dagger}$ and use $A = \sum_j \lambda_j P_j$.

Common pitfalls

- Do not use A^T in place of $A^{\dagger}$ for complex matrices.
- Do not forget which slot of the inner product is conjugate-linear.

- Remember $(\alpha A)^\dagger = \overline{\alpha} A^\dagger$.
- Remember $(AB)^\dagger = B^\dagger A^\dagger$.
- A real spectrum alone does not imply that an operator is Hermitian.
- Diagonalizability is not the same as unitary diagonalizability.
- Match the order of eigenvalues in D to the columns of U.
- Vectors in the same degenerate eigenspace are not automatically orthogonal.

Core formulas

$$\left\langle T^\dagger \mathbf{u}, \mathbf{v} \right\rangle = \langle \mathbf{u}, T\mathbf{v} \rangle, \qquad A^\dagger = \overline{A}^T.$$

$$H^\dagger = H, \qquad K^\dagger = -K, \qquad U^\dagger U = I.$$

$$H^\dagger = H \Rightarrow \lambda \in \mathbb{R}, \qquad K^\dagger = -K \Rightarrow \lambda \in i\mathbb{R}, \qquad U^\dagger U = I \Rightarrow |\lambda| = 1.$$

$$U^\dagger A U = D, \qquad A = U D U^\dagger, \qquad P_j = \mathbf{u}_j \mathbf{u}_j^\dagger.$$

$$A = \sum_j \lambda_j P_j, \qquad f(A) = \sum_j f(\lambda_j) P_j, \qquad e^{At} = \sum_j e^{\lambda_j t} P_j.$$

13 Take-home assignment: five problems

Exceptional weekly assignment — due Tuesday, October 13, 2026

Because this lecture is being held online, the usual Thursday assignment is replaced this week by the five problems below.
Submit one organized solution set at the beginning of class on **Tuesday, October 13, 2026**. Show all intermediate calculations, state the properties you use, and include every requested verification. Answers without supporting work will not be considered complete.
The problems are intentionally connected to the main ideas of Lecture 05. No solutions are included in these notes.

Problem 1 — Adjoint and operator classification

For each matrix below, compute the adjoint and classify the matrix as Hermitian, anti-Hermitian, unitary, or none of these. A matrix may belong to more than one class. Justify every classification.

$$A = \begin{pmatrix} 3 & 2-i \\ 2+i & -1 \end{pmatrix}, \qquad B = \begin{pmatrix} 2i & 1+i \\ -1+i & -i \end{pmatrix},$$

$$C = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}, \qquad D = \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}.$$

For every matrix classified as unitary, verify explicitly that it preserves the norm of

$$\mathbf{v} = \begin{pmatrix} 1+i \\ 2 \end{pmatrix}.$$

Problem 2 — Complete application of the Spectral Theorem

Consider

$$H = \begin{pmatrix} 3 & 1+i \\ 1-i & 3 \end{pmatrix}.$$

(a) Verify that H is Hermitian.

(b) Compute its characteristic polynomial and eigenvalues.

(c) Find and normalize an eigenvector for each eigenvalue.

(d) Verify the orthogonality of the normalized eigenvectors using the complex inner product.

(e) Construct U and D and verify $U^\dagger U = I$.

(f) Verify both $U^\dagger H U = D$ and $H = U D U^\dagger$.

(g) Construct the spectral projectors and write $H = \sum_j \lambda_j P_j$.

Problem 3 — Degeneracy, Gram–Schmidt, and projectors

Let

$$M = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{pmatrix},$$

and consider

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \qquad \mathbf{v}_2 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \qquad \mathbf{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

(a) Identify all eigenvalues, their algebraic multiplicities, and their eigenspaces.

(b) Apply Gram–Schmidt to $\mathbf{v}_1, \mathbf{v}_2$ inside the degenerate eigenspace.

(c) Complete the resulting vectors to an orthonormal eigenbasis of $\mathbb{R}^3$.

(d) Construct the projector P_2 onto the full eigenspace of $\lambda = 2$ and the projector P_5 onto the eigenspace of $\lambda = 5$.

(e) Verify $P_2 + P_5 = I$, $P_2 P_5 = 0$, and

$$M = 2P_2 + 5P_5.$$

(f) Explain why P_2 does not depend on the particular orthonormal basis chosen inside the degenerate eigenspace.

Problem 4 — Functions of operators and unitary evolution

Use the matrix

$$H = \begin{pmatrix} 2 & i \\ -i & 2 \end{pmatrix}.$$

You may use its eigenvalues and projectors from Example 5.5 and Example 5.6.

(a) Compute H^2 using its spectral projectors.

(b) Find H^{-1} without direct matrix inversion.

(c) Obtain a closed expression for e^{-iHt}.

(d) Verify that e^{-iHt} is unitary for every real t.

(e) Evaluate e^{-iH0} and

$$\frac{d}{dt} e^{-iHt}\Big|_{t=0}.$$

(f) For $\psi(0) = \frac{1}{\sqrt{2}}(1,1)^T$, compute $\psi(t) = e^{-iHt}\psi(0)$ and verify directly that $\|\psi(t)\| = 1$.

Problem 5 — Anisotropic plasma transport and the Hall term

Let $\widehat{\mathbf{b}} = (0,0,1)^T$ and define

$$P_\parallel = \widehat{\mathbf{b}}\widehat{\mathbf{b}}^{,T}, \qquad P_\perp = I - P_\parallel.$$

The conductivity tensor is

$$\boldsymbol{\sigma} = \begin{pmatrix} \sigma_\perp & -\sigma_H & 0 \\ \sigma_H & \sigma_\perp & 0 \\ 0 & 0 & \sigma_\parallel \end{pmatrix}.$$

(a) Compute $P_\parallel$ and $P_\perp$ explicitly and verify that they are orthogonal projectors.

(b) Decompose $\boldsymbol{\sigma}$ into its symmetric and antisymmetric parts,

$$\boldsymbol{\sigma}_{\mathrm{s}} = \frac{1}{2}(\boldsymbol{\sigma} + \boldsymbol{\sigma}^T), \qquad \boldsymbol{\sigma}_{\mathrm{a}} = \frac{1}{2}(\boldsymbol{\sigma} - \boldsymbol{\sigma}^T).$$

(c) Show that

$$\boldsymbol{\sigma}_{\mathrm{s}} = \sigma_{\parallel} P_{\parallel} + \sigma_{\perp} P_{\perp}.$$

(d) Find the eigenvalues and eigenspaces of $\boldsymbol{\sigma}_{\mathrm{s}}$ and identify any degeneracy.

(e) For a real electric field $\mathbf{E}$, show that

$$\mathbf{E}^{T} \boldsymbol{\sigma}_{\mathrm{a}} \mathbf{E} = 0.$$

Interpret this result in relation to the Hall contribution and explain why the real orthogonal spectral decomposition above applies to the symmetric part rather than to the full tensor.