GES-207-4 — Mathematical Methods of Physics

National Institute for Space Research (INPE)
Graduate Program in Space Geophysics (PPGGES)

Lecture Notes 02

Inner Product, Norm, and Orthogonality

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Learning goals and study scope. In Lecture 1 we developed the algebraic structure of vector spaces and linear operators. In this lecture we add geometric structure. By the end of the lecture, you should be able to define and use inner products in real and complex vector spaces; obtain norms and distances from an inner product; apply the Cauchy–Schwarz and triangle inequalities; work with orthogonal and orthonormal sets; construct orthogonal projections; interpret orthogonal decompositions and Bessel's inequality; and apply these ideas to the geometry of magnetic fields relative to an interplanetary shock.

1. Part I: From Vector Spaces to Inner Products

1.1. From Vector Spaces to Geometry

In Lecture 1 we studied the linear structure of a vector space. Vector addition and scalar multiplication allowed us to define linear combinations, linear independence, bases, dimension, linear transformations, kernels, and images. This structure alone, however, does not provide a notion of *geometry*.

In an abstract vector space V, we can form

$$\alpha\mathbf{u} + \beta\mathbf{v},$$

but without additional structure we cannot answer questions such as: What is the length of a vector? What is the distance between two vectors? When are two vectors perpendicular? What is the angle between two directions? What is the component of one vector along another?

In $\mathbb{R}^n$, these questions are answered using the Euclidean dot product

$$\mathbf{u}\cdot\mathbf{v} = \sum_{i=1}^{n} u_i v_i.$$

For example,

$$\|\mathbf{v}\| = \sqrt{\mathbf{v}\cdot\mathbf{v}},$$

and $\mathbf{u}$ and $\mathbf{v}$ are orthogonal when $\mathbf{u}\cdot\mathbf{v} = 0$.

The central goal of this lecture is to generalize this structure to much broader vector spaces, including spaces of complex vectors, matrices, and functions. This generalization is the **inner product**.

Central idea. A vector space provides the structure required for *linear algebra*. An inner product adds *geometry*, making it possible to define length, distance, orthogonality, angle, and projection.

This distinction is especially important in mathematical physics. A vector need not be an arrow in $\mathbb{R}^3$. Functions, fields, and matrices may also form vector spaces. We may therefore ask whether two functions are orthogonal, or what the distance is between two signals or two solutions of a differential equation. Such questions require an appropriate inner product.

1.2. Bilinear and Sesquilinear Forms

Before defining an inner product, it is useful to consider more general maps that take two vectors and return a scalar.

Definition 2.1: Bilinear Form

Let V be a vector space over $\mathbb{R}$. A **bilinear form** is a map

$$B : V \times V \longrightarrow \mathbb{R}$$

that is linear in each argument:

$$B(\alpha\mathbf{u} + \beta\mathbf{v}, \mathbf{w}) = \alpha B(\mathbf{u}, \mathbf{w}) + \beta B(\mathbf{v}, \mathbf{w}), \tag{1}$$

$$B(\mathbf{u}, \alpha\mathbf{v} + \beta\mathbf{w}) = \alpha B(\mathbf{u}, \mathbf{v}) + \beta B(\mathbf{u}, \mathbf{w}). \tag{2}$$

A bilinear form is **symmetric** if

$$B(\mathbf{u}, \mathbf{v}) = B(\mathbf{v}, \mathbf{u}).$$

Not every symmetric bilinear form is an inner product. For example, in $\mathbb{R}^2$,

$$B(\mathbf{u}, \mathbf{v}) = u_1 v_1 - u_2 v_2$$

is symmetric and bilinear, but

$$B((0,1),(0,1)) = -1.$$

It therefore cannot define a squared length. Positivity will be essential.

1.2.1. The complex case

A direct use of transpose is insufficient in complex spaces. For example, with

$$\mathbf{v} = \begin{pmatrix} 1 \\ i \end{pmatrix},$$

we obtain

$$\mathbf{v}^T \mathbf{v} = 1 + i^2 = 0,$$

even though $\mathbf{v} \neq \mathbf{0}$.

The appropriate operation is the **Hermitian transpose**

$$\mathbf{u}^H = \overline{\mathbf{u}}^T.$$

Then

$$\mathbf{v}^H \mathbf{v} = |1|^2 + |i|^2 = 2 > 0.$$

This motivates sesquilinear forms.

Definition 2.2: Sesquilinear Form

Let V be a vector space over $\mathbb{C}$. A map

$$S : V \times V \longrightarrow \mathbb{C}$$

is **sesquilinear**, with the convention adopted in this course, if it is conjugate-linear in the first argument and linear in the second:

$$S(\alpha\mathbf{u} + \beta\mathbf{v}, \mathbf{w}) = \overline{\alpha}\, S(\mathbf{u}, \mathbf{w}) + \overline{\beta}\, S(\mathbf{v}, \mathbf{w}), \tag{3}$$

$$S(\mathbf{u}, \alpha\mathbf{v} + \beta\mathbf{w}) = \alpha S(\mathbf{u}, \mathbf{v}) + \beta S(\mathbf{u}, \mathbf{w}). \tag{4}$$

Convention used throughout the course. Inner products are always taken to be conjugate-linear in the **first** argument and linear in the **second**:

$$\langle \alpha\mathbf{u}, \mathbf{v} \rangle = \overline{\alpha} \langle \mathbf{u}, \mathbf{v} \rangle, \qquad \langle \mathbf{u}, \alpha\mathbf{v} \rangle = \alpha \langle \mathbf{u}, \mathbf{v} \rangle.$$

Some texts adopt the opposite convention. Both are mathematically equivalent, but one convention must be used consistently.

1.3. Inner Product

Definition 2.3: Inner Product

Let V be a vector space over $\mathbb{K}$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. An **inner product** is a map

$$\langle \cdot, \cdot \rangle : V \times V \longrightarrow \mathbb{K}$$

such that, for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ and $\alpha, \beta \in \mathbb{K}$:

1. **Conjugate symmetry**
$$\langle \mathbf{u}, \mathbf{v} \rangle = \overline{\langle \mathbf{v}, \mathbf{u} \rangle}.$$
2. **Conjugate-linearity in the first argument**
$$\langle \alpha \mathbf{u} + \beta \mathbf{v}, \mathbf{w} \rangle = \overline{\alpha} \langle \mathbf{u}, \mathbf{w} \rangle + \overline{\beta} \langle \mathbf{v}, \mathbf{w} \rangle.$$
3. **Positivity**
$$\langle \mathbf{v}, \mathbf{v} \rangle \geq 0.$$
4. **Positive definiteness**
$$\langle \mathbf{v}, \mathbf{v} \rangle = 0 \quad \Longleftrightarrow \quad \mathbf{v} = \mathbf{0}.$$

Linearity in the second argument follows from conjugate symmetry and conjugate-linearity in the first:

$$\langle \mathbf{u}, \alpha \mathbf{v} + \beta \mathbf{w} \rangle = \alpha \langle \mathbf{u}, \mathbf{v} \rangle + \beta \langle \mathbf{u}, \mathbf{w} \rangle.$$

For $\mathbb{K} = \mathbb{R}$, complex conjugation has no effect and the inner product is simply symmetric and bilinear.

Pre-Hilbert and Hilbert spaces. A vector space equipped with an inner product is called an **inner-product space**, or a **pre-Hilbert space**. A **Hilbert space** is an inner-product space that is complete with respect to the distance induced by its inner product.

1.4. Fundamental Examples

Example 2.1: Euclidean Inner Product in $\mathbb{R}^n$

For $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$,

$$\boxed{\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u}^T \mathbf{v} = \sum_{j=1}^{n} u_j v_j.}$$

This is the familiar Euclidean dot product.

Example 2.2: Standard Inner Product in $\mathbb{C}^n$

For $\mathbf{u}, \mathbf{v} \in \mathbb{C}^n$,

$$\boxed{\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u}^H \mathbf{v} = \sum_{j=1}^{n} \overline{u_j} v_j.}$$

In particular,

$$\langle \mathbf{v}, \mathbf{v} \rangle = \sum_{j=1}^{n} |v_j|^2 \geq 0,$$

with equality only for $\mathbf{v} = \mathbf{0}$.

Example 2.3: Inner Product in a Function Space

For square-integrable functions on $[a, b]$,

$$\boxed{\langle f, g \rangle = \int_a^b \overline{f(x)}\, g(x)\, dx.}$$

Then

$$\langle f, f \rangle = \int_a^b |f(x)|^2\, dx \geq 0.$$

In $L^2([a, b])$, equality to zero means that $f = 0$ almost everywhere. This inner product is central to Fourier analysis, quantum mechanics, signal processing, and the theory of differential equations.

1.5. The Frobenius Inner Product

Matrices themselves form vector spaces. For $A, B \in \mathcal{M}_{m\times n}(\mathbb{C})$, define

$$\boxed{\langle A, B \rangle_F = \operatorname{tr}(A^H B) = \sum_{i=1}^{m} \sum_{j=1}^{n} \overline{A_{ij}} B_{ij}.}$$

This is the **Frobenius inner product**. Conjugate symmetry follows from

$$\overline{\operatorname{tr}(B^H A)} = \operatorname{tr}\Big((B^H A)^H\Big) = \operatorname{tr}(A^H B).$$

Moreover,

$$\langle A, A \rangle_F = \sum_{i,j} |A_{ij}|^2 \geq 0,$$

with equality only when $A = O$. Hence

$$\boxed{\|A\|_F = \sqrt{\operatorname{tr}(A^H A)} = \left(\sum_{i,j} |A_{ij}|^2 \right)^{1/2}.}$$

Vectors, functions, and matrices look very different, but the same abstract inner-product structure applies to all of them. This is one of the main advantages of the language of linear algebra: a single theorem can apply to many physically distinct objects.

2. Part II: Norm, Distance, and the Cauchy–Schwarz Inequality

2.1. Norm Induced by an Inner Product

Definition 2.4: Induced Norm

If V is an inner-product space, the norm induced by the inner product is

$$\boxed{\|\mathbf{v}\| = \sqrt{\langle \mathbf{v}, \mathbf{v}\rangle}.}$$

The basic norm properties follow from the inner-product axioms. Positivity and definiteness are immediate. For any scalar α,

$$\begin{aligned} \|\alpha\mathbf{v}\|^2 &= \langle \alpha\mathbf{v}, \alpha\mathbf{v}\rangle && (5)\\ &= |\alpha|^2\langle \mathbf{v}, \mathbf{v}\rangle, && (6) \end{aligned}$$

and therefore

$$\boxed{\|\alpha\mathbf{v}\| = |\alpha|\,\|\mathbf{v}\|.}$$

The triangle inequality will follow from Cauchy–Schwarz.

2.2. Distance Between Vectors

Definition 2.5: Distance Induced by a Norm

For $\mathbf{u}, \mathbf{v} \in V$,

$$\boxed{d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\|.}$$

In $\mathbb{R}^n$, this gives the ordinary Euclidean distance. In $L^2([a,b])$,

$$d(f,g) = \left(\int_a^b |f(x) - g(x)|^2\, dx\right)^{1/2}.$$

Thus two functions may be interpreted as close when their difference is small in the L^2 sense.

The logical hierarchy is

$$\boxed{\text{inner product} \longrightarrow \text{norm} \longrightarrow \text{distance.}}$$

Each step introduces a new geometric concept from the preceding structure.

2.3. The Cauchy–Schwarz Inequality

Theorem 2.1: Cauchy–Schwarz Inequality

For all $\mathbf{u}, \mathbf{v}$ in an inner-product space,

$$\boxed{|\langle \mathbf{u}, \mathbf{v}\rangle| \leq \|\mathbf{u}\|\,\|\mathbf{v}\|.}$$

Equality holds if and only if $\mathbf{u}$ and $\mathbf{v}$ are linearly dependent.

2.3.1. Proof

If $\mathbf{v} = \mathbf{0}$, the result is immediate. Assume $\mathbf{v} \neq \mathbf{0}$ and define

$$\mathbf{w} = \mathbf{u} - \frac{\langle \mathbf{v}, \mathbf{u}\rangle}{\|\mathbf{v}\|^2}\mathbf{v}.$$

Since $\|\mathbf{w}\|^2 \geq 0$,

$$0 \leq \left\langle \mathbf{u} - \frac{\langle \mathbf{v}, \mathbf{u}\rangle}{\|\mathbf{v}\|^2}\mathbf{v}, \mathbf{u} - \frac{\langle \mathbf{v}, \mathbf{u}\rangle}{\|\mathbf{v}\|^2}\mathbf{v} \right\rangle \tag{7}$$

$$= \|\mathbf{u}\|^2 - \frac{|\langle \mathbf{u}, \mathbf{v}\rangle|^2}{\|\mathbf{v}\|^2}. \tag{8}$$

Therefore

$$|\langle \mathbf{u}, \mathbf{v}\rangle|^2 \leq \|\mathbf{u}\|^2 \|\mathbf{v}\|^2,$$

and taking square roots gives

$$|\langle \mathbf{u}, \mathbf{v}\rangle| \leq \|\mathbf{u}\|\,\|\mathbf{v}\|.$$

Equality occurs precisely when $\mathbf{w} = \mathbf{0}$, which is equivalent to linear dependence. □

2.4. Triangle Inequality

Theorem 2.2: Triangle Inequality

For all $\mathbf{u}, \mathbf{v} \in V$,

$$\boxed{\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|.}$$

Indeed,

$$\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + 2\,\mathrm{Re}\langle \mathbf{u}, \mathbf{v}\rangle + \|\mathbf{v}\|^2 \tag{9}$$

$$\leq \|\mathbf{u}\|^2 + 2|\langle \mathbf{u}, \mathbf{v}\rangle| + \|\mathbf{v}\|^2 \tag{10}$$

$$\leq \|\mathbf{u}\|^2 + 2\|\mathbf{u}\|\,\|\mathbf{v}\| + \|\mathbf{v}\|^2 \tag{11}$$

$$= (\|\mathbf{u}\| + \|\mathbf{v}\|)^2. \tag{12}$$

Taking the nonnegative square root proves the result.

2.5. Angle Between Vectors

In a **real** inner-product space, for nonzero vectors $\mathbf{u}, \mathbf{v}$, define

$$\boxed{\cos\theta = \frac{\langle \mathbf{u}, \mathbf{v}\rangle}{\|\mathbf{u}\|\,\|\mathbf{v}\|}, \qquad 0 \le \theta \le \pi.}$$

Cauchy–Schwarz guarantees that the right-hand side lies in $[-1, 1]$.

In complex inner-product spaces there is no unique direct analogue of the oriented Euclidean angle. A commonly used convention is

$$\cos\theta = \frac{|\langle \mathbf{u}, \mathbf{v}\rangle|}{\|\mathbf{u}\|\,\|\mathbf{v}\|}, \qquad 0 \le \theta \le \frac{\pi}{2}.$$

Orthogonality, however, remains unambiguous in both real and complex spaces.

2.6. Example in a Function Space

Example 2.4: Cauchy–Schwarz and Angle in L^2

Consider

$$f(x) = 1, \qquad g(x) = x, \qquad x \in [0, 1],$$

with

$$\langle f, g\rangle = \int_0^1 f(x)g(x)\,dx.$$

Then

$$\langle f, g\rangle = \int_0^1 x\,dx = \frac{1}{2}.$$

The norms are

$$\|f\| = \left(\int_0^1 1\,dx\right)^{1/2} = 1,$$

and

$$\|g\| = \left(\int_0^1 x^2\,dx\right)^{1/2} = \frac{1}{\sqrt{3}}.$$

Thus Cauchy–Schwarz gives

$$\frac{1}{2} \le \frac{1}{\sqrt{3}}.$$

Since the functions are real,

$$\cos\theta = \frac{1/2}{1/\sqrt{3}} = \frac{\sqrt{3}}{2},$$

so

$$\boxed{\theta = \frac{\pi}{6} = 30^\circ.}$$

This example illustrates that geometric concepts such as length and angle extend naturally to functions. This viewpoint will become fundamental in Fourier analysis.

3. Part III: Orthogonality, Projections, and Decomposition

3.1. Orthogonality

Definition 2.6: Orthogonality

Two vectors $\mathbf{u}, \mathbf{v} \in V$ are **orthogonal** if

$$\boxed{\langle \mathbf{u}, \mathbf{v} \rangle = 0.}$$

We write $\mathbf{u} \perp \mathbf{v}$.

Example 2.5: Orthogonal Functions

On $[-1, 1]$, let

$$f(x) = 1, \qquad g(x) = x.$$

Then

$$\langle f, g \rangle = \int_{-1}^{1} x\, dx = 0.$$

Hence $f \perp g$. More generally, on an interval symmetric about the origin, the integral of an odd integrand vanishes. This parity property is often a simple way to recognize orthogonality between functions.

3.2. Orthogonal and Orthonormal Sets

Definition 2.7: Orthogonal and Orthonormal Sets

A set $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ is **orthogonal** if

$$\langle \mathbf{v}_i, \mathbf{v}_j \rangle = 0 \qquad (i \neq j).$$

It is **orthonormal** if, in addition,

$$\|\mathbf{v}_j\| = 1$$

for every j. Equivalently,

$$\boxed{\langle \mathbf{v}_i, \mathbf{v}_j \rangle = \delta_{ij}.}$$

Any orthogonal set of nonzero vectors can be normalized by replacing $\mathbf{v}_j$ with

$$\mathbf{u}_j = \frac{\mathbf{v}_j}{\|\mathbf{v}_j\|}.$$

3.3. Orthogonality and Linear Independence

Proposition 2.1: Orthogonal Nonzero Vectors Are Linearly Independent

Every orthogonal set of nonzero vectors is linearly independent.

Suppose

$$\alpha_1 \mathbf{v}_1 + \cdots + \alpha_k \mathbf{v}_k = \mathbf{0}.$$

Taking the inner product with $\mathbf{v}_j$ gives

$$\alpha_j \|\mathbf{v}_j\|^2 = 0.$$

Since $\mathbf{v}_j \neq \mathbf{0}$, we have $\|\mathbf{v}_j\|^2 > 0$, so $\alpha_j = 0$. This holds for every j. Therefore all coefficients vanish. □

Consequently, in an n-dimensional space, any set of n nonzero mutually orthogonal vectors forms a basis. After normalization, it becomes an orthonormal basis.

3.4. Pythagorean Theorem

Theorem 2.3: Pythagorean Theorem

If $\mathbf{u} \perp \mathbf{v}$, then

$$\boxed{\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2.}$$

Indeed,

$$\begin{aligned} \|\mathbf{u} + \mathbf{v}\|^2 &= \langle \mathbf{u} + \mathbf{v}, \mathbf{u} + \mathbf{v} \rangle & (13) \\ &= \|\mathbf{u}\|^2 + \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{v}, \mathbf{u} \rangle + \|\mathbf{v}\|^2, & (14) \end{aligned}$$

and the cross terms vanish.

More generally, if $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ is orthonormal, then

$$\boxed{\left\| \sum_{j=1}^{k} \alpha_j \mathbf{u}_j \right\|^2 = \sum_{j=1}^{k} |\alpha_j|^2.}$$

3.5. Orthogonal Projection onto a Direction

Let $\hat{\mathbf{n}}$ be a unit vector. We seek the component of $\mathbf{v}$ along $\hat{\mathbf{n}}$. Write

$$\mathbf{v}_\parallel = \alpha \hat{\mathbf{n}}, \qquad \mathbf{v}_\perp = \mathbf{v} - \alpha \hat{\mathbf{n}},$$

and require the residual to be orthogonal to $\hat{\mathbf{n}}$:

$$\langle \hat{\mathbf{n}}, \mathbf{v}_\perp \rangle = 0.$$

Therefore

$$\langle \hat{\mathbf{n}}, \mathbf{v} \rangle - \alpha \langle \hat{\mathbf{n}}, \hat{\mathbf{n}} \rangle = 0.$$

Since $\|\hat{\mathbf{n}}\| = 1$,

$$\alpha = \langle \hat{\mathbf{n}}, \mathbf{v} \rangle.$$

Hence

$$\boxed{\mathbf{v}_{\|} = \langle \hat{\mathbf{n}}, \mathbf{v} \rangle \hat{\mathbf{n}},}$$

and

$$\boxed{\mathbf{v}_{\perp} = \mathbf{v} - \langle \hat{\mathbf{n}}, \mathbf{v} \rangle \hat{\mathbf{n}}.}$$

3.5.1. Connection with the projection operator from Lecture 1

In $\mathbb{R}^n$, define

$$P = \hat{\mathbf{n}} \hat{\mathbf{n}}^T.$$

Then

$$P\mathbf{v} = (\hat{\mathbf{n}} \cdot \mathbf{v}) \hat{\mathbf{n}} = \mathbf{v}_{\|},$$

and

$$(I - P)\mathbf{v} = \mathbf{v}_{\perp}.$$

Moreover,

$$P^2 = \hat{\mathbf{n}} (\hat{\mathbf{n}}^T \hat{\mathbf{n}}) \hat{\mathbf{n}}^T = P.$$

Thus P is idempotent. In a complex vector space, the corresponding orthogonal projector is

$$P = \hat{\mathbf{n}} \hat{\mathbf{n}}^H.$$

An arbitrary idempotent operator need not define an *orthogonal* projection. For the projector above, symmetry (or Hermiticity in the complex case) provides the orthogonal geometry.

3.6. Projection onto a Subspace

Let

$$W = \operatorname{span}\{\mathbf{u}_1, \dots, \mathbf{u}_k\},$$

where $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ is an orthonormal basis of W. The orthogonal projection of $\mathbf{v}$ onto W is

$$\boxed{P_W \mathbf{v} = \sum_{j=1}^{k} \langle \mathbf{u}_j, \mathbf{v} \rangle \mathbf{u}_j.}$$

To see why, define

$$\mathbf{r} = \mathbf{v} - P_W \mathbf{v}.$$

For each i,

$$\begin{aligned} \langle \mathbf{u}_i, \mathbf{r} \rangle &= \langle \mathbf{u}_i, \mathbf{v} \rangle - \sum_{j=1}^{k} \langle \mathbf{u}_j, \mathbf{v} \rangle \langle \mathbf{u}_i, \mathbf{u}_j \rangle & (15) \\ &= \langle \mathbf{u}_i, \mathbf{v} \rangle - \langle \mathbf{u}_i, \mathbf{v} \rangle & (16) \\ &= 0. & (17) \end{aligned}$$

Therefore the residual is orthogonal to every vector in W.

3.7. Orthogonal Complement

Definition 2.8: Orthogonal Complement

For a subset $S \subseteq V$, the **orthogonal complement** is

$$\boxed{S^{\perp} = \{\mathbf{v} \in V : \langle \mathbf{u}, \mathbf{v} \rangle = 0 \text{ for every } \mathbf{u} \in S\}.}$$

If W is a finite-dimensional subspace of an inner-product space, then

$$\boxed{V = W \oplus W^{\perp}.}$$

Every vector therefore admits a unique decomposition

$$\mathbf{v} = \mathbf{v}_W + \mathbf{v}_{W^{\perp}}.$$

In a Hilbert space, the same orthogonal-decomposition result extends to every **closed** subspace W. Closedness is essential in the infinite-dimensional setting.

3.8. Bessel's Identity and Inequality

Let $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ be orthonormal and let

$$P_W \mathbf{v} = \sum_{j=1}^{k} \langle \mathbf{u}_j, \mathbf{v} \rangle \mathbf{u}_j.$$

Since

$$\mathbf{v} = P_W \mathbf{v} + (\mathbf{v} - P_W \mathbf{v})$$

is an orthogonal decomposition, Pythagoras gives

$$\|\mathbf{v}\|^2 = \|P_W \mathbf{v}\|^2 + \|\mathbf{v} - P_W \mathbf{v}\|^2.$$

Because the basis is orthonormal,

$$\|P_W \mathbf{v}\|^2 = \sum_{j=1}^{k} |\langle \mathbf{u}_j, \mathbf{v} \rangle|^2.$$

Therefore

$$\boxed{\|\mathbf{v}\|^2 = \sum_{j=1}^{k} |\langle \mathbf{u}_j, \mathbf{v}\rangle|^2 + \|\mathbf{v} - P_W\mathbf{v}\|^2.}$$

Since the last term is nonnegative,

$$\boxed{\sum_{j=1}^{k} |\langle \mathbf{u}_j, \mathbf{v}\rangle|^2 \leq \|\mathbf{v}\|^2.}$$

This is **Bessel's inequality**.

The quantities $|\langle \mathbf{u}_j, \mathbf{v}\rangle|^2$ may be interpreted as the contributions of the orthonormal directions to the squared norm, or "energy", of $\mathbf{v}$. In a complete orthonormal expansion in a Hilbert space, this idea leads to Parseval's identity and is fundamental to Fourier analysis.

4. Part IV: Application to Space Geophysics

4.1. Magnetic-Field Decomposition Relative to a Shock

Consider a shock surface and let $\hat{\mathbf{n}}$ be its unit normal. Let $\mathbf{B}$ be the local magnetic field. The field can be decomposed into components parallel and perpendicular to the shock normal:

$$\boxed{\mathbf{B} = \mathbf{B}_{\parallel} + \mathbf{B}_{\perp}.}$$

The parallel component is

$$\boxed{\mathbf{B}_{\parallel} = \langle \hat{\mathbf{n}}, \mathbf{B}\rangle \hat{\mathbf{n}},}$$

and the perpendicular component is

$$\boxed{\mathbf{B}_{\perp} = \mathbf{B} - \mathbf{B}_{\parallel}.}$$

By construction,

$$\langle \hat{\mathbf{n}}, \mathbf{B}_{\perp}\rangle = 0.$$

Since the two components are orthogonal,

$$\boxed{\|\mathbf{B}\|^2 = \|\mathbf{B}_{\parallel}\|^2 + \|\mathbf{B}_{\perp}\|^2.}$$

4.2. The Angle θ_{Bn}

The angle between the upstream magnetic field and the shock normal is commonly denoted by θ_{Bn}. Because the normal may be represented with either orientation, it is convenient to use

$$\boxed{\cos\theta_{Bn} = \frac{|\langle \hat{\mathbf{n}}, \mathbf{B}\rangle|}{\|\mathbf{B}\|},}$$

where $\|\hat{\mathbf{n}}\| = 1$. Thus

$$\boxed{\theta_{Bn} = \arccos\left(\frac{|\langle \hat{\mathbf{n}}, \mathbf{B}\rangle|}{\|\mathbf{B}\|}\right), \qquad 0 \le \theta_{Bn} \le 90^\circ.}$$

Cauchy–Schwarz guarantees that

$$|\langle \hat{\mathbf{n}}, \mathbf{B}\rangle| \le \|\mathbf{B}\|,$$

so the argument of the inverse cosine lies in $[0, 1]$.

Because

$$\|\mathbf{B}_\parallel\| = |\langle \hat{\mathbf{n}}, \mathbf{B}\rangle|,$$

we may also write

$$\cos\theta_{Bn} = \frac{\|\mathbf{B}_\parallel\|}{\|\mathbf{B}\|}, \qquad \sin\theta_{Bn} = \frac{\|\mathbf{B}_\perp\|}{\|\mathbf{B}\|}.$$

4.3. Quasi-Parallel and Quasi-Perpendicular Shocks

A commonly used geometrical classification is

$$\theta_{Bn} < 45^\circ \quad \Rightarrow \quad \text{quasi-parallel shock,}$$

and

$$\theta_{Bn} > 45^\circ \quad \Rightarrow \quad \text{quasi-perpendicular shock.}$$

The value 45° is the conventional boundary between the two regimes.

This geometry affects the interaction of charged particles with collisionless shocks, including reflection, scattering, transport, and acceleration processes. The classification is therefore not merely geometric: it helps organize physically distinct shock environments.

4.4. Numerical Example

Example 2.6: Magnetic Field Relative to a Shock Normal

Let

$$\mathbf{B} = \begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix} \text{ nT}, \qquad \hat{\mathbf{n}} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

Then

$$\langle \hat{\mathbf{n}}, \mathbf{B}\rangle = 3 \text{ nT},$$

and therefore

$$\mathbf{B}_\parallel = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} \text{ nT}, \qquad \mathbf{B}_\perp = \begin{pmatrix} 0 \\ 4 \\ 0 \end{pmatrix} \text{ nT}.$$

Their norms are

$$\|\mathbf{B}_{\|}\| = 3 \text{ nT}, \qquad \|\mathbf{B}_{\perp}\| = 4 \text{ nT}, \qquad \|\mathbf{B}\| = 5 \text{ nT}.$$

Pythagoras gives

$$5^2 = 3^2 + 4^2.$$

The shock angle is

$$\theta_{Bn} = \arccos\left(\frac{3}{5}\right) \approx 53.1^\circ.$$

Hence the shock is **quasi-perpendicular**.

This example displays the complete conceptual chain:

$$\boxed{\text{inner product} \longrightarrow \text{norm} \longrightarrow \text{projection} \longrightarrow \text{orthogonality} \longrightarrow \text{angle.}}$$

4.5. Operator Formulation

In $\mathbb{R}^3$, define

$$P_{\|} = \hat{\mathbf{n}}\hat{\mathbf{n}}^T, \qquad P_{\perp} = I - P_{\|}.$$

Then

$$\mathbf{B}_{\|} = P_{\|}\mathbf{B}, \qquad \mathbf{B}_{\perp} = P_{\perp}\mathbf{B}.$$

The projectors satisfy

$$P_{\|}^2 = P_{\|}, \qquad P_{\perp}^2 = P_{\perp}, \qquad P_{\|}P_{\perp} = O.$$

For the numerical example above,

$$P_{\|} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad P_{\perp} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

This formulation explicitly connects the algebra of linear operators from Lecture 1 with the geometry introduced in this lecture.

5. Lecture Summary

In this lecture we added geometric structure to the vector spaces studied previously. The inner product generalizes the Euclidean dot product and provides a unified language for real and complex vectors, functions, matrices, and other mathematical objects.

From the inner product we defined the norm

$$\|\mathbf{v}\| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle},$$

and the distance

$$d(\mathbf{u},\mathbf{v}) = \|\mathbf{u}-\mathbf{v}\|.$$

The Cauchy–Schwarz inequality,

$$|\langle \mathbf{u},\mathbf{v}\rangle| \le \|\mathbf{u}\|\,\|\mathbf{v}\|,$$

links the inner product and norm and supports both the triangle inequality and the definition of angle in real inner-product spaces.

Orthogonality is characterized by

$$\langle \mathbf{u},\mathbf{v}\rangle = 0.$$

It leads to the Pythagorean theorem, linear independence of nonzero orthogonal sets, orthogonal complements, and projections.

For a unit direction $\hat{\mathbf{n}}$,

$$P_{\|}\mathbf{v} = \langle \hat{\mathbf{n}},\mathbf{v}\rangle\hat{\mathbf{n}},$$

while

$$P_{\perp}\mathbf{v} = \mathbf{v} - P_{\|}\mathbf{v}.$$

More generally, if $\{\mathbf{u}_1,\ldots,\mathbf{u}_k\}$ is an orthonormal basis of a subspace W,

$$P_W\mathbf{v} = \sum_{j=1}^{k}\langle \mathbf{u}_j,\mathbf{v}\rangle\mathbf{u}_j.$$

The associated orthogonal decomposition yields Bessel's inequality,

$$\sum_{j=1}^{k}|\langle \mathbf{u}_j,\mathbf{v}\rangle|^2 \le \|\mathbf{v}\|^2,$$

which anticipates orthonormal expansions and Fourier analysis.

Finally, the same ideas were applied to the decomposition of a magnetic field relative to a shock normal and to the definition of the shock angle θ_{Bn}.

Conceptual structure of the lecture.

$$\boxed{\begin{array}{c} \text{Vector space} \\ \downarrow \\ \text{Inner product} \\ \downarrow \\ \text{Norm and distance} \\ \downarrow \\ \text{Cauchy–Schwarz} \\ \downarrow \\ \text{Angle and orthogonality} \\ \downarrow \\ \text{Projections} \\ \downarrow \\ \text{Orthogonal decompositions} \\ \downarrow \\ \text{Orthonormal bases and Bessel} \end{array}}$$

The inner product is therefore the structure that transforms a purely algebraic vector space into a space in which geometry can be developed.

6. Part V: Further Study — When Does a Norm Come from an Inner Product?

Further study. Every inner product determines a norm through

$$\|\mathbf{v}\| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}.$$

A natural converse question is:

$$\boxed{\text{Is every norm induced by some inner product?}}$$

The answer is no. This part gives the exact criterion that distinguishes norms that arise from inner products.

6.1. The Parallelogram Law

Theorem 2.4: Parallelogram Law

If a norm is induced by an inner product, then for all $\mathbf{u}, \mathbf{v} \in V$,

$$\boxed{\|\mathbf{u}+\mathbf{v}\|^2 + \|\mathbf{u}-\mathbf{v}\|^2 = 2\|\mathbf{u}\|^2 + 2\|\mathbf{v}\|^2.}$$

Indeed,

$$\|\mathbf{u}+\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + 2\,\mathrm{Re}\langle \mathbf{u}, \mathbf{v}\rangle + \|\mathbf{v}\|^2, \tag{18}$$

$$\|\mathbf{u}-\mathbf{v}\|^2 = \|\mathbf{u}\|^2 - 2\,\mathrm{Re}\langle \mathbf{u}, \mathbf{v}\rangle + \|\mathbf{v}\|^2. \tag{19}$$

Adding the equations cancels the cross terms and proves the identity.

Geometrically, $\mathbf{u}+\mathbf{v}$ and $\mathbf{u}-\mathbf{v}$ are the diagonals of the parallelogram generated by $\mathbf{u}$ and $\mathbf{v}$. The theorem states that the sum of the squares of the diagonal lengths equals the sum of the squares of the lengths of the four sides.

6.2. Not Every Norm Comes from an Inner Product

Example 2.7: The ℓ^1 Norm

On $\mathbb{R}^2$, consider

$$\|\mathbf{x}\|_1 = |x_1| + |x_2|.$$

Take

$$\mathbf{u} = (1, 0), \qquad \mathbf{v} = (0, 1).$$

Then

$$\|\mathbf{u}\|_1 = \|\mathbf{v}\|_1 = 1,$$

whereas

$$\|\mathbf{u}+\mathbf{v}\|_1 = \|\mathbf{u}-\mathbf{v}\|_1 = 2.$$

Hence

$$\|\mathbf{u}+\mathbf{v}\|_1^2 + \|\mathbf{u}-\mathbf{v}\|_1^2 = 8,$$

but

$$2\|\mathbf{u}\|_1^2 + 2\|\mathbf{v}\|_1^2 = 4.$$

The parallelogram law fails. Therefore no inner product on $\mathbb{R}^2$ induces the norm $\|\cdot\|_1$.

Every inner-product space is naturally a normed space:

$$\text{inner product} \quad \Longrightarrow \quad \text{norm}.$$

The converse is false in general:

$$\text{norm} \quad \not\Longrightarrow \quad \text{inner product.}$$

An inner-product space therefore carries more structure than a general normed space.

6.3. Polarization Identity

When a norm is induced by an inner product, the norm contains enough information to reconstruct the inner product.

Theorem 2.5: Polarization Identity

In a real inner-product space,

$$\boxed{\langle \mathbf{u}, \mathbf{v} \rangle = \frac{1}{4}\left(\|\mathbf{u}+\mathbf{v}\|^2 - \|\mathbf{u}-\mathbf{v}\|^2\right).}$$

This follows by subtracting

$$\|\mathbf{u}-\mathbf{v}\|^2 = \|\mathbf{u}\|^2 - 2\langle \mathbf{u}, \mathbf{v} \rangle + \|\mathbf{v}\|^2$$

from

$$\|\mathbf{u}+\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + 2\langle \mathbf{u}, \mathbf{v} \rangle + \|\mathbf{v}\|^2.$$

Complex case. With the convention used in this course—conjugate-linear in the first argument and linear in the second—the polarization identity is

$$\boxed{\begin{aligned}\langle \mathbf{u}, \mathbf{v} \rangle = \frac{1}{4}\Big(&\|\mathbf{u}+\mathbf{v}\|^2 - \|\mathbf{u}-\mathbf{v}\|^2 \\ &- i\|\mathbf{u}+i\mathbf{v}\|^2 + i\|\mathbf{u}-i\mathbf{v}\|^2\Big).\end{aligned}}$$

Thus, in both the real and complex cases, an inner product can be completely reconstructed from its induced norm.

6.4. Jordan–von Neumann Theorem

Theorem 2.6: Jordan–von Neumann

A norm $\|\cdot\|$ on a real or complex vector space is induced by an inner product if and only if it satisfies the parallelogram law:

$$\boxed{\|\mathbf{u}+\mathbf{v}\|^2 + \|\mathbf{u}-\mathbf{v}\|^2 = 2\|\mathbf{u}\|^2 + 2\|\mathbf{v}\|^2}$$

for every $\mathbf{u}, \mathbf{v} \in V$.

When this condition holds, the corresponding inner product is uniquely determined by the norm through the polarization identity.

The forward implication has already been proved. For the converse, the polarization formula is used to define a candidate inner product, and the parallelogram law supplies the relations required to verify its linearity and the remaining inner-product axioms.

Therefore,

$$\boxed{\text{norm induced by an inner product} \quad \Longleftrightarrow \quad \text{parallelogram law.}}$$

6.5. Geometric Interpretation

A norm assigns a length to each vector:

$$\mathbf{v} \longmapsto \|\mathbf{v}\|.$$

An inner product contains richer information because it compares pairs of vectors:

$$(\mathbf{u}, \mathbf{v}) \longmapsto \langle \mathbf{u}, \mathbf{v} \rangle.$$

This additional information naturally produces orthogonality, angles, and projections. The parallelogram law identifies exactly those norms that carry enough geometric information for an inner product to be reconstructed.

Summary of the further study.

$$\boxed{\text{inner product} \Longrightarrow \text{induced norm} \Longrightarrow \text{parallelogram law.}}$$

The converse is also true:

$$\boxed{\text{parallelogram law} \Longrightarrow \text{a unique inner product.}}$$

The polarization identity provides that inner product explicitly.

7. Preparation Problems

Preparation Problems — Lectures 1 and 2 (Presentation Next Tuesday)

Instructions. Solve both problems using the concepts developed in this week's lectures. Your solutions should be ready for the next class, on Tuesday.

At the beginning of the next class, two students will be selected to present the solutions, one student for each problem. Each presentation will last approximately 5 minutes and should emphasize the reasoning, the main steps of the solution, and the interpretation of the result.

1. **Problem 1 — Linear Transformations, Kernel, and Image**

   Consider the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$, defined by $T(\mathbf{x}) = A\mathbf{x}$, where

   $$A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 4 & 3 \\ 0 & 0 & 1 \end{pmatrix}.$$

   (a) Compute $\det(A)$, explicitly showing the procedure used. Is A invertible?

   (b) Use Gaussian elimination to determine $\operatorname{rank}(A)$. Then use the Rank–Nullity Theorem to determine $\dim(\ker A)$ and explicitly find a basis for $\ker(A)$.

   (c) Consider

   $$A\mathbf{x} = \mathbf{b}, \qquad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}.$$

   Using the Rouché–Capelli theorem, determine the condition that b_1, b_2, b_3 must satisfy for the system to have a solution.

   (d) Interpret the previous result in terms of the image of T. Determine $\dim(\operatorname{Im} T)$ and find a basis for $\operatorname{Im} T$.

2. **Problem 2 — Inner Product, Orthogonality, and Projection**

   Consider $\mathbb{R}^3$ with the Euclidean inner product and

   $$\mathbf{B} = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}, \qquad \mathbf{n} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

   (a) Determine $\|\mathbf{B}\|$ and normalize $\mathbf{n}$ to obtain

   $$\hat{\mathbf{n}} = \frac{\mathbf{n}}{\|\mathbf{n}\|}.$$

   (b) Compute the orthogonal projection of $\mathbf{B}$ onto the direction defined by $\hat{\mathbf{n}}$,

   $$\mathbf{B}_{\|} = \langle \hat{\mathbf{n}}, \mathbf{B} \rangle \hat{\mathbf{n}},$$

   and determine

   $$\mathbf{B}_{\perp} = \mathbf{B} - \mathbf{B}_{\|}.$$

   (c) Verify explicitly that

   $$\langle \mathbf{B}_{\|}, \mathbf{B}_{\perp} \rangle = 0$$

and use the Pythagorean theorem to verify

$$\|\mathbf{B}\|^2 = \|\mathbf{B}_\parallel\|^2 + \|\mathbf{B}_\perp\|^2.$$

(d) Determine

$$\theta_{Bn} = \arccos\left(\frac{|\langle \hat{\mathbf{n}}, \mathbf{B}\rangle|}{\|\mathbf{B}\|}\right).$$

Using 45° as the conventional boundary, classify the geometry as quasi-parallel or quasi-perpendicular.

(e) Construct

$$P_\parallel = \hat{\mathbf{n}}\hat{\mathbf{n}}^T$$

and verify that

$$P_\parallel^2 = P_\parallel.$$

Then show that

$$P_\parallel \mathbf{B} = \mathbf{B}_\parallel.$$

Briefly explain how this result connects the concept of a linear operator from Lecture 1 with the geometry introduced in Lecture 2.