GES-207-4 — Mathematical Methods of Physics

National Institute for Space Research (INPE)
Graduate Program in Space Geophysics (PPGGES)

Lecture Notes 01

Matrices and Vector Spaces

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Learning goals. This lecture develops the algebraic foundations used throughout the course. The main goals are to work fluently with matrices and linear systems; relate determinant, invertibility, rank, kernel, and image; understand vector spaces, subspaces, bases, and linear operators; distinguish active transformations from passive changes of basis; introduce the dual space and covectors; and review the cross product in three-dimensional Euclidean space.

1. Part I: Fundamentals of Matrices

1.1. Definitions and Notation

A matrix over a field $\mathbb{K}$ is a rectangular array of scalars. We write

$$A = (a_{ij}) \in \mathcal{M}_{m\times n}(\mathbb{K}),$$

where m is the number of rows and n the number of columns. Two matrices are equal when they have the same dimensions and corresponding entries are equal.

Matrix addition and scalar multiplication are defined componentwise:

$$(A+B)_{ij} = a_{ij} + b_{ij}, \qquad (\alpha A)_{ij} = \alpha a_{ij}.$$

If $A \in \mathcal{M}_{m\times n}$ and $B \in \mathcal{M}_{n\times p}$, their product is

$$C = AB \in \mathcal{M}_{m\times p}, \qquad c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}.$$

Thus each entry of AB is the inner product of a row of A with a column of B. Matrix multiplication is associative and distributive, but in general it is not commutative:

$$AB \neq BA.$$

The transpose of A is

$$(A^T)_{ij} = a_{ji},$$

and for complex matrices the Hermitian transpose is

$$A^H = \overline{A}^T.$$

Useful identities include

$$(AB)^T = B^T A^T, \qquad (AB)^H = B^H A^H.$$

For a square matrix $A \in \mathcal{M}_{n\times n}$, the trace is

$$\boxed{\mathrm{tr}(A) = \sum_{i=1}^{n} a_{ii}.}$$

It is linear and satisfies

$$\mathrm{tr}(AB) = \mathrm{tr}(BA)$$

whenever the products are defined and square.

1.1.1. Matrix Multiplication in Detail

The dimensional condition in matrix multiplication is essential. If

$$A \in \mathcal{M}_{m\times n}(\mathbb{K}), \qquad B \in \mathcal{M}_{r\times p}(\mathbb{K}),$$

then AB exists only when $n = r$. The result then belongs to $\mathcal{M}_{m\times p}(\mathbb{K})$.

Writing the ith row of A as $\mathbf{r}_i^T$ and the jth column of B as $\mathbf{c}_j$, we have

$$(AB)_{ij} = \mathbf{r}_i^T \mathbf{c}_j.$$

Thus matrix multiplication combines rows and columns through scalar products. This is different from saying that the matrix product itself is an inner product: AB is generally another matrix.

For example,

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 3 & 2 \end{pmatrix}, \qquad B = \begin{pmatrix} 2 & 1 \\ -1 & 0 \\ 3 & 4 \end{pmatrix}.$$

Then

$$AB = \begin{pmatrix} 1(2) + 2(-1) + (-1)3 & 1(1) + 2(0) + (-1)4 \\ 0(2) + 3(-1) + 2(3) & 0(1) + 3(0) + 2(4) \end{pmatrix} = \begin{pmatrix} -3 & -3 \\ 3 & 8 \end{pmatrix}.$$

The noncommutativity of matrix multiplication has a direct interpretation for linear transformations. If A and B represent two transformations, then

$$AB\mathbf{x} = A(B\mathbf{x})$$

means that B acts first and A acts second. Reversing the order generally changes the physical or geometrical operation.

1.1.2. Trace and Cyclic Permutations

For a square matrix,

$$\mathrm{tr}(A) = \sum_i A_{ii}.$$

The trace is linear:

$$\mathrm{tr}(\alpha A + \beta B) = \alpha\,\mathrm{tr}(A) + \beta\,\mathrm{tr}(B).$$

A particularly useful identity is

$$\mathrm{tr}(AB) = \mathrm{tr}(BA).$$

Indeed,

$$\mathrm{tr}(AB) = \sum_i (AB)_{ii} = \sum_{i,j} A_{ij}B_{ji} = \sum_{j,i} B_{ji}A_{ij} = \mathrm{tr}(BA).$$

More generally, the trace is invariant under cyclic permutations:

$$\mathrm{tr}(ABC) = \mathrm{tr}(BCA) = \mathrm{tr}(CAB).$$

This does *not* mean that the factors may be arbitrarily permuted.

The trace is also invariant under a change of basis. If

$$A' = S^{-1}AS,$$

then

$$\mathrm{tr}(A') = \mathrm{tr}(S^{-1}AS) = \mathrm{tr}(ASS^{-1}) = \mathrm{tr}(A).$$

Hence the trace is a property of the linear operator, not of a particular matrix representation.

1.2. Special Matrices

The zero matrix O has all entries equal to zero. The identity matrix

$$I_n = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$$

satisfies

$$AI_n = I_nA = A.$$

A diagonal matrix has nonzero entries only on its main diagonal. A real matrix is symmetric if

$$A^T = A,$$

and skew-symmetric if

$$A^T = -A.$$

A complex matrix is Hermitian if

$$A^H = A.$$

Orthogonal and unitary matrices satisfy, respectively,

$$Q^T Q = I, \qquad U^H U = I.$$

These matrices preserve Euclidean or Hermitian lengths and angles.

Every real square matrix can be decomposed uniquely as

$$A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T),$$

that is, into symmetric and skew-symmetric parts.

1.3. Determinant and Inverse

For a square matrix A, the determinant is a scalar that encodes important information about the associated linear transformation. For a 2×2 matrix,

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad \det(A) = ad - bc.$$

For larger matrices, the determinant can be evaluated by cofactor expansion:

$$\det(A) = \sum_{j=1}^{n} a_{ij} C_{ij}, \qquad C_{ij} = (-1)^{i+j} M_{ij},$$

where M_{ij} is the corresponding minor.

Important properties are

$$\det(AB) = \det(A)\det(B), \qquad \det(A^T) = \det(A),$$

and, when A is invertible,

$$\det(A^{-1}) = \frac{1}{\det(A)}.$$

Invertibility Criterion

For $A \in \mathcal{M}_{n\times n}(\mathbb{K})$,

$$\boxed{A \text{ is invertible} \quad \Longleftrightarrow \quad \det(A) \neq 0.}$$

If the inverse exists, it is unique and satisfies

$$AA^{-1} = A^{-1}A = I_n.$$

Geometrically, $|\det(A)|$ is the factor by which the associated linear map scales oriented n-dimensional volume. A zero determinant means that the transformation collapses at least one dimension.

1.3.1. Permutation Definition and Cofactor Expansion

For

$$A = (A_{ij}) \in \mathcal{M}_{n\times n}(\mathbb{K}),$$

the determinant may be defined by

$$\boxed{\det(A) = \sum_{\sigma\in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} A_{i,\sigma(i)},}$$

where S_n is the set of all permutations of $\{1,\ldots,n\}$ and $\operatorname{sgn}(\sigma)$ is $+1$ for even permutations and -1 for odd permutations.

For practical calculations, the cofactor expansion is often more useful. Removing row i and column j from A gives the minor matrix associated with A_{ij}. Its determinant is the minor M_{ij}, and

$$C_{ij} = (-1)^{i+j} M_{ij}$$

is the corresponding cofactor. Expanding along any fixed row i,

$$\boxed{\det(A) = \sum_{j=1}^{n} A_{ij}C_{ij}.}$$

Similarly, expansion along a column j gives

$$\det(A) = \sum_{i=1}^{n} A_{ij}C_{ij}.$$

For

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix},$$

expansion along the first row gives

$$\det(A) = a(ei - fh) - b(di - fg) + c(dh - eg).$$

This is a cofactor expansion. The familiar Sarrus mnemonic is restricted to 3×3 matrices and should not be confused with the general definition.

1.3.2. Effect of Elementary Row Operations

The determinant reacts predictably to elementary row operations:

1. Interchanging two rows changes the sign of the determinant.
2. Multiplying one row by α multiplies the determinant by α.
3. Adding a multiple of one row to another leaves the determinant unchanged.

These rules make Gaussian elimination an efficient determinant algorithm. For a triangular matrix,

$$\boxed{\det(A) = \prod_{i=1}^{n} A_{ii}.}$$

1.3.3. Adjugate and the Inverse

The cofactor matrix is

$$C = (C_{ij}),$$

and the adjugate is

$$\mathrm{adj}(A) = C^T.$$

One can show that

$$A\ \mathrm{adj}(A) = \mathrm{adj}(A)\, A = \det(A) I.$$

Therefore, if $\det(A) \neq 0$,

$$\boxed{A^{-1} = \frac{1}{\det(A)}\,\mathrm{adj}(A).}$$

For

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

this becomes

$$\boxed{A^{-1} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.}$$

1.3.4. Gauss–Jordan Inversion

A computationally systematic way to obtain the inverse is to form the augmented matrix

$$[A|I]$$

and apply elementary row operations. If A is invertible, the reduction produces

$$[A|I] \longrightarrow [I|A^{-1}].$$

If the left block cannot be reduced to the identity, then A is singular.

Example 1.1: Conductance Matrix of a Resistive Network

Linear systems arise naturally in network models. If node potentials are collected in a vector $\mathbf{V}$ and injected currents in $\mathbf{I}$, a linear resistive network can be written as

$$G\mathbf{V} = \mathbf{I},$$

where G is the conductance matrix. For example,

$$G = \begin{pmatrix} g_1 + g_3 & -g_3 \\ -g_3 & g_2 + g_3 \end{pmatrix}.$$

Its determinant is

$$\det(G) = (g_1 + g_3)(g_2 + g_3) - g_3^2 = g_1 g_2 + g_1 g_3 + g_2 g_3.$$

For positive conductances this determinant is positive, so G is invertible and the node potentials are uniquely determined:

$$\mathbf{V} = G^{-1}\mathbf{I}.$$

This simple example illustrates how invertibility becomes a statement about the uniqueness of a physical solution.

1.4. Linear Systems and Rank

A linear system can be written compactly as

$$A\mathbf{x} = \mathbf{b}.$$

Gaussian elimination uses elementary row operations to reduce the augmented matrix $[A|\mathbf{b}]$ to an equivalent echelon form.

The rank of a matrix is the number of pivot positions in an echelon form:

$$\boxed{\operatorname{rank}(A) = \dim(\text{column space of } A).}$$

Rouché–Capelli Theorem

The system

$$A\mathbf{x} = \mathbf{b}$$

is consistent if and only if

$$\boxed{\operatorname{rank}(A) = \operatorname{rank}([A|\mathbf{b}]).}$$

If the system is consistent and $\operatorname{rank}(A) = n$, the solution is unique. If it is consistent and $\text{rank}(A) < n$, there are free variables and infinitely many solutions.

For the homogeneous system

$$A\mathbf{x} = \mathbf{0},$$

the solution set is always a vector subspace. For a nonhomogeneous consistent system, the solution set is generally an *affine* set:

$$\mathbf{x} = \mathbf{x}_p + \mathbf{x}_h, \qquad \mathbf{x}_h \in \ker(A).$$

1.4.1. Rank from Row and Column Spaces

The row rank and column rank of a matrix are equal. Their common value is the rank:

$$\text{rank}(A) = \dim(\text{Row}\, A) = \dim(\text{Col}\, A).$$

Although row operations change the individual rows, they preserve the rank. This is why echelon form is useful.

For

$$A \in \mathcal{M}_{m\times n},$$

$$0 \le \text{rank}(A) \le \min(m, n).$$

A square $n \times n$ matrix has full rank when

$$\text{rank}(A) = n.$$

For square matrices, the following statements are equivalent:

$$\boxed{\det(A) \neq 0 \iff A^{-1} \text{ exists} \iff \text{rank}(A) = n \iff \ker(A) = \{\mathbf{0}\}.}$$

Example 1.2: Equilibrium of Forces at a Node

Suppose unknown force components are collected in $\mathbf{x} = (x_1, x_2, x_3)^T$ and equilibrium gives

$$A\mathbf{x} = \mathbf{b}.$$

The augmented matrix $[A|\mathbf{b}]$ contains both the geometry of the force directions and the applied load. Gaussian elimination may reveal a row

$$(0 \quad 0 \quad 0 \,|\, c).$$

If $c \neq 0$, the equilibrium equations are incompatible. If $c = 0$, the row is redundant. Thus Rouché–Capelli separates two different questions: whether a solution exists and, if it exists, whether it is unique.

For a consistent nonhomogeneous system, the set of solutions is not usually a vector subspace because it need not contain the zero vector. If $\mathbf{x}_p$ is one particular solution, every solution has the form

$$\mathbf{x} = \mathbf{x}_p + \mathbf{z}, \qquad \mathbf{z} \in \ker(A).$$

The solution set is therefore an affine translate of the null space.

2. Part II: Vector Spaces, Dual Space, and Cross Product

2.1. Vector Spaces and Subspaces

Vector Space

A vector space V over a field $\mathbb{K}$ is a set equipped with vector addition and scalar multiplication satisfying the usual closure, associativity, commutativity, identity, inverse, and distributive axioms.

Standard examples include

$$\mathbb{R}^n, \qquad \mathbb{C}^n, \qquad \mathcal{M}_{m\times n}(\mathbb{K}),$$

as well as spaces of functions, polynomials, and solutions of homogeneous linear differential equations.

A subset $W \subseteq V$ is a subspace if it contains $\mathbf{0}$ and is closed under linear combinations:

$$\mathbf{u}, \mathbf{v} \in W, \quad \alpha, \beta \in \mathbb{K} \quad \Longrightarrow \quad \alpha\mathbf{u} + \beta\mathbf{v} \in W.$$

A linear combination of vectors $\mathbf{v}_1, \dots, \mathbf{v}_k$ is

$$\alpha_1\mathbf{v}_1 + \cdots + \alpha_k\mathbf{v}_k.$$

Their span is the set of all such combinations:

$$\operatorname{span}\{\mathbf{v}_1, \dots, \mathbf{v}_k\}.$$

The vectors are linearly independent when

$$\alpha_1\mathbf{v}_1 + \cdots + \alpha_k\mathbf{v}_k = \mathbf{0}$$

implies

$$\alpha_1 = \cdots = \alpha_k = 0.$$

A basis is a linearly independent spanning set. Every vector then has a unique coordinate representation in that basis.

2.1.1. The Vector-Space Axioms

For completeness, the operations in a vector space V over $\mathbb{K}$ satisfy, for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ and $\alpha, \beta \in \mathbb{K}$:

$$\begin{aligned} \mathbf{u} + \mathbf{v} &= \mathbf{v} + \mathbf{u}, \\ (\mathbf{u} + \mathbf{v}) + \mathbf{w} &= \mathbf{u} + (\mathbf{v} + \mathbf{w}), \\ \mathbf{v} + \mathbf{0} &= \mathbf{v}, \\ \mathbf{v} + (-\mathbf{v}) &= \mathbf{0}, \\ \alpha(\mathbf{u} + \mathbf{v}) &= \alpha\mathbf{u} + \alpha\mathbf{v}, \\ (\alpha + \beta)\mathbf{v} &= \alpha\mathbf{v} + \beta\mathbf{v}, \\ (\alpha\beta)\mathbf{v} &= \alpha(\beta\mathbf{v}), \\ 1\mathbf{v} &= \mathbf{v}. \end{aligned}$$

These axioms deliberately do not specify what the elements of V "look like." This abstraction allows the same linear-algebraic reasoning to apply to vectors, matrices, functions, polynomials, and fields.

2.1.2. Examples from Mathematical Physics

The set

$$\mathcal{P}_n = \{a_0 + a_1 x + \cdots + a_n x^n\}$$

is a vector space of dimension $n + 1$. A natural basis is

$$\{1, x, \ldots, x^n\}.$$

The set of all solutions of a homogeneous linear differential equation, such as

$$y'' + \omega^2 y = 0,$$

is also a vector space. In this example,

$$y(x) = A\cos(\omega x) + B\sin(\omega x),$$

so the solution space has basis

$$\{\cos(\omega x), \sin(\omega x)\}$$

and dimension two.

By contrast, the solutions of

$$y'' + \omega^2 y = f(x), \qquad f \neq 0,$$

do not generally form a vector space. Their set is affine: a particular solution plus the vector space of homogeneous solutions.

2.2. Linear Operators

Linear Transformation

A map

$$T : V \longrightarrow W$$

is linear if

$$T(\alpha\mathbf{u} + \beta\mathbf{v}) = \alpha T(\mathbf{u}) + \beta T(\mathbf{v}).$$

The kernel and image are

$$\ker(T) = \{\mathbf{v} \in V : T(\mathbf{v}) = \mathbf{0}\},$$

and

$$\mathrm{Im}(T) = \{T(\mathbf{v}) : \mathbf{v} \in V\}.$$

Both are vector subspaces.

Rank–Nullity Theorem

If V is finite-dimensional and $T : V \to W$ is linear, then

$$\boxed{\dim(V) = \dim(\ker T) + \dim(\mathrm{Im}\, T).}$$

For a matrix transformation $T(\mathbf{x}) = A\mathbf{x}$,

$$\dim(\mathrm{Im}\, T) = \mathrm{rank}(A).$$

An operator $P : V \to V$ satisfying

$$P^2 = P$$

is called idempotent. Such an operator provides the algebraic decomposition

$$\mathbf{v} = P\mathbf{v} + (I - P)\mathbf{v},$$

with

$$P\mathbf{v} \in \mathrm{Im}(P), \qquad (I - P)\mathbf{v} \in \ker(P).$$

In fact,

$$V = \mathrm{Im}(P) \oplus \ker(P).$$

Orthogonal projections will acquire their geometric meaning after an inner product is introduced in Lecture 2.

2.2.1. Matrix Representation of a Linear Operator

Let

$$\mathcal{B} = \{\mathbf{e}_1, \dots, \mathbf{e}_n\}$$

be a basis of V and

$$\mathcal{C} = \{\mathbf{f}_1, \dots, \mathbf{f}_m\}$$

a basis of W. A linear transformation $T : V \to W$ is completely determined by its action on the basis vectors:

$$T(\mathbf{e}_j) = \sum_{i=1}^{m} A_{ij}\mathbf{f}_i.$$

The coefficients A_{ij} form the matrix

$$[T]_{\mathcal{C}\leftarrow\mathcal{B}} = A.$$

For

$$\mathbf{v} = v^j\mathbf{e}_j,$$

linearity gives

$$T(\mathbf{v}) = v^j T(\mathbf{e}_j) = A_{ij}v^j\mathbf{f}_i.$$

Thus the coordinate columns obey

$$\boxed{[T(\mathbf{v})]_{\mathcal{C}} = [T]_{\mathcal{C}\leftarrow\mathcal{B}}[\mathbf{v}]_{\mathcal{B}}.}$$

Composition of linear transformations corresponds to matrix multiplication. If $S : U \to V$ and $T : V \to W$, then

$$[T \circ S] = [T][S],$$

with compatible choices of bases. This explains why the order of matrix multiplication matters.

2.2.2. Kernel, Image, Injectivity, and Surjectivity

The kernel measures the directions annihilated by a transformation:

$$\ker(T) = \{\mathbf{v} : T\mathbf{v} = \mathbf{0}\}.$$

The image contains all attainable outputs:

$$\operatorname{Im}(T) = \{T\mathbf{v} : \mathbf{v} \in V\}.$$

A linear map is injective exactly when

$$\ker(T) = \{\mathbf{0}\},$$

and it is surjective exactly when

$$\mathrm{Im}(T) = W.$$

For $T : V \to V$ in finite dimension, injectivity, surjectivity, and invertibility are equivalent.

2.2.3. Why Rank–Nullity Is a Dimension Balance

Suppose

$$\{\mathbf{k}_1, \dots, \mathbf{k}_r\}$$

is a basis of $\ker(T)$ and extend it to a basis of V:

$$\{\mathbf{k}_1, \dots, \mathbf{k}_r, \mathbf{v}_{r+1}, \dots, \mathbf{v}_n\}.$$

The nonzero information carried by T is represented by

$$T(\mathbf{v}_{r+1}), \dots, T(\mathbf{v}_n),$$

which form a basis of $\mathrm{Im}(T)$. Hence

$$\dim(\mathrm{Im}\, T) = n - r,$$

so

$$\boxed{\dim(\ker T) + \dim(\mathrm{Im}\, T) = \dim V.}$$

The theorem expresses a conservation of dimension: directions lost to the kernel cannot simultaneously contribute independent directions to the image.

2.3. Change of Basis

Let $\mathcal{B} = \{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ and $\mathcal{B}' = \{\mathbf{e}'_1, \dots, \mathbf{e}'_n\}$ be two bases of V. The vector itself is independent of the basis, but its coordinate column changes:

$$\mathbf{v} = \sum_i v^i \mathbf{e}_i = \sum_i v'^i \mathbf{e}'_i.$$

If the columns of S are the coordinates of the new basis vectors expressed in the old basis,

$$[\mathbf{e}'_1 \ \cdots \ \mathbf{e}'_n] = [\mathbf{e}_1 \ \cdots \ \mathbf{e}_n] S,$$

then the coordinate columns satisfy

$$\boxed{[\mathbf{v}]_{\mathcal{B}} = S[\mathbf{v}]_{\mathcal{B}'}, \qquad [\mathbf{v}]_{\mathcal{B}'} = S^{-1}[\mathbf{v}]_{\mathcal{B}}.}$$

If $T : V \to V$ has matrix A in $\mathcal{B}$, then its matrix in $\mathcal{B}'$ is

$$\boxed{A' = S^{-1} A S.}$$

Matrices related by this transformation are similar and represent the same linear operator in different bases.

Active versus passive transformations. An *active* transformation changes the geometric vector:

$$\mathbf{v} \longmapsto T\mathbf{v}.$$

A *passive* transformation changes the basis used to describe the same vector. Keeping these two viewpoints distinct prevents many common errors in coordinate transformations.

2.3.1. Rotation of a Vector versus Rotation of the Basis

The distinction between active and passive transformations is worth making explicit. In an active rotation, the basis is fixed while the vector changes:

$$[\mathbf{v}']_{\mathcal{B}} = R[\mathbf{v}]_{\mathcal{B}}.$$

In a passive rotation, the geometric vector is fixed while the basis changes. If the new basis is related to the old one by the same rotation R, the coordinates transform inversely:

$$[\mathbf{v}]_{\mathcal{B}'} = R^{-1}[\mathbf{v}]_{\mathcal{B}}.$$

For an orthogonal rotation matrix,

$$R^{-1} = R^T.$$

The similarity of these formulas is a common source of sign and transpose errors, so the physical question "what is actually being changed?" should always be answered first.

Example 1.3: Rotation of Coordinates in the Plane

Let

$$R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

An active counterclockwise rotation gives

$$\mathbf{v}' = R(\theta)\mathbf{v}.$$

If instead the coordinate axes are rotated counterclockwise by θ while the vector remains fixed, then

$$[\mathbf{v}]_{\mathcal{B}'} = R(-\theta)[\mathbf{v}]_{\mathcal{B}} = R(\theta)^T[\mathbf{v}]_{\mathcal{B}}.$$

2.3.2. Coordinate Systems in Space Geophysics

Space-geophysics data are routinely represented in different coordinate systems. A magnetic-field vector may be expressed in geographic, geomagnetic, solar-magnetospheric, or solar-ecliptic coordinates. The physical field does not change merely because the basis changes:

$$\mathbf{B} = B^i \mathbf{e}_i = B'^i \mathbf{e}'_i.$$

Only its components change. The same principle applies to velocity, electric-field, and current-density vectors. Coordinate transformations are therefore applications of change-of-basis algebra, not changes in the underlying physical quantity.

2.4. Dual Space and Covectors

The dual space V^* is the vector space of linear functionals

$$\omega : V \longrightarrow \mathbb{K}.$$

Its elements are called **covectors**.

Given a basis

$$\mathcal{B} = \{\mathbf{e}_1, \dots, \mathbf{e}_n\},$$

the dual basis

$$\mathcal{B}^* = \{\mathbf{e}^1, \dots, \mathbf{e}^n\}$$

is defined by

$$\boxed{\mathbf{e}^i(\mathbf{e}_j) = \delta^i{}_j.}$$

If

$$\mathbf{v} = v^j \mathbf{e}_j, \qquad \omega = \omega_i \mathbf{e}^i,$$

then

$$\boxed{\omega(\mathbf{v}) = \omega_i v^i.}$$

Under a basis change, vector components and covector components transform in opposite ways. This is the origin of the terms *contravariant* and *covariant* components.

A particularly important physical example comes from a scalar field Φ. Its differential

$$d\Phi$$

is naturally a covector: it maps a displacement vector to the directional change of Φ. In Euclidean space, the metric identifies this covector with the familiar gradient vector:

$$\nabla\Phi = (d\Phi)^\sharp.$$

This distinction becomes essential in curvilinear coordinates, differential geometry, and relativity.

2.5. Cross Product in $\mathbb{R}^3$

For

$$\mathbf{a} = (a_1, a_2, a_3), \qquad \mathbf{b} = (b_1, b_2, b_3),$$

the cross product is

$$\boxed{\mathbf{a} \times \mathbf{b} = \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix}.}$$

Equivalently,

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}.$$

Its main properties are

$$\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a},$$

$$\mathbf{a} \cdot (\mathbf{a} \times \mathbf{b}) = 0, \qquad \mathbf{b} \cdot (\mathbf{a} \times \mathbf{b}) = 0,$$

and

$$\|\mathbf{a} \times \mathbf{b}\| = \|\mathbf{a}\| \, \|\mathbf{b}\| \sin\theta.$$

Thus the magnitude equals the area of the parallelogram generated by $\mathbf{a}$ and $\mathbf{b}$.

The scalar triple product

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$$

gives the oriented volume of the corresponding parallelepiped.

Physical Example: Lorentz Magnetic Force

The magnetic part of the Lorentz force is

$$\mathbf{F}_B = q\,\mathbf{v} \times \mathbf{B}.$$

It is perpendicular to both $\mathbf{v}$ and $\mathbf{B}$. Therefore

$$\mathbf{F}_B \cdot \mathbf{v} = 0,$$

so the magnetic force does no mechanical work on the particle. It can change the direction of the velocity, but not the particle's kinetic energy by itself.

The familiar cross product is special to oriented three-dimensional Euclidean geometry. In more general settings, the fundamental object is the exterior product

$$\mathbf{a} \wedge \mathbf{b},$$

a bivector. In three dimensions, the metric and orientation allow the Hodge dual to identify this bivector with a vector:

$$\mathbf{a} \times \mathbf{b} = *(\mathbf{a} \wedge \mathbf{b}).$$

2.5.1. Levi–Civita Symbol

The three-dimensional cross product can be written compactly using the Levi–Civita symbol:

$$\boxed{(\mathbf{a} \times \mathbf{b})_i = \varepsilon_{ijk} a_j b_k.}$$

Here

$$\varepsilon_{ijk} = \begin{cases} +1, & (i, j, k) \text{ is an even permutation of } (1, 2, 3), \\ -1, & (i, j, k) \text{ is an odd permutation of } (1, 2, 3), \\ 0, & \text{two indices are equal.} \end{cases}$$

Useful contraction identities include

$$\varepsilon_{ijk}\varepsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km},$$

from which many vector identities follow. For example,

$$\begin{aligned} \|\mathbf{a} \times \mathbf{b}\|^2 &= \varepsilon_{ijk} a_j b_k \varepsilon_{imn} a_m b_n \\ &= (\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}) a_j b_k a_m b_n \\ &= \|\mathbf{a}\|^2 \|\mathbf{b}\|^2 - (\mathbf{a} \cdot \mathbf{b})^2. \end{aligned}$$

Therefore,

$$\|\mathbf{a} \times \mathbf{b}\| = \|\mathbf{a}\| \, \|\mathbf{b}\| \sin\theta.$$

The determinant of a 3×3 matrix can also be expressed as

$$\det(A) = \varepsilon_{ijk} A_{1i} A_{2j} A_{3k}.$$

This makes the relation between determinants, orientation, volume, and the cross product explicit.

2.5.2. Torque and Lorentz Force

Two standard physical examples are

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$$

for torque and

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

for the Lorentz force.

For the magnetic contribution,

$$\mathbf{F}_B = q\,\mathbf{v} \times \mathbf{B},$$

$$\mathbf{F}_B \cdot \mathbf{v} = 0.$$

Hence

$$\frac{d}{dt}\left(\frac{1}{2}mv^2\right) = \mathbf{F}_B \cdot \mathbf{v} = 0.$$

A static magnetic field changes the direction of the particle velocity but does not change its kinetic energy.

Example 1.5: Cyclotron Motion

Let

$$\mathbf{B} = B\hat{\mathbf{z}}, \qquad \mathbf{E} = \mathbf{0}.$$

The equation of motion is

$$m\frac{d\mathbf{v}}{dt} = q\,\mathbf{v} \times \mathbf{B}.$$

Writing

$$\mathbf{v} = (v_x, v_y, v_z),$$

we obtain

$$m\dot{v}_x = qBv_y, \qquad m\dot{v}_y = -qBv_x, \qquad m\dot{v}_z = 0.$$

Differentiating the first equation once more gives

$$\ddot{v}_x + \Omega_c^2 v_x = 0, \qquad \Omega_c = \frac{|q|B}{m}.$$

The perpendicular velocity therefore rotates with cyclotron frequency Ω_c, while v_z remains constant. The trajectory is a helix around the magnetic-field direction. The gyroradius is

$$\boxed{\rho_c = \frac{mv_\perp}{|q|B}.}$$

This example connects the algebraic cross product directly to charged particle motion in space plasmas.

2.5.3. Why the Cross Product Is Special to Three Dimensions

The cross product combines two vectors and returns a vector perpendicular to both. This construction relies on the special relationship between two-dimensional oriented area elements and vectors in three dimensions.

The more fundamental construction is the exterior product

$$\mathbf{a} \wedge \mathbf{b},$$

which is a bivector representing an oriented area element. In an oriented three-dimensional Euclidean space, the Hodge star maps bivectors to vectors:

$$\boxed{\mathbf{a} \times \mathbf{b} = *(\mathbf{a} \wedge \mathbf{b}).}$$

Thus the familiar cross product already contains metric and orientation information. The exterior product generalizes naturally to arbitrary dimensions.

3. Lecture Summary

The lecture developed the algebraic framework on which the geometric structures of later lectures will be built. Matrices represent linear maps; determinant and rank diagnose invertibility and dimensional collapse; kernels and images describe what a transformation annihilates and what it can produce; bases provide coordinates without changing the underlying vector; dual vectors act linearly on vectors; and the three-dimensional cross product encodes an oriented perpendicular direction.

The central relationships are

$$\boxed{\begin{array}{c}\text{Matrices and linear systems}\\ \downarrow\\ \text{Vector spaces and subspaces}\\ \downarrow\\ \text{Linear transformations}\\ \downarrow\\ \ker(T)\ \text{and}\ \operatorname{Im}(T)\\ \downarrow\\ \text{Bases and coordinate changes}\\ \downarrow\\ \text{Dual space and covectors}\\ \downarrow\\ \text{Three-dimensional vector geometry}\end{array}}$$

The next step is to add an inner product. That additional structure will make length, distance, angle, orthogonality, and orthogonal projection meaningful in an abstract vector space.